THE GENERICITY CONJECTURE

Sy D. Friedman*
MIT

The Genericity Conjecture, as stated in Beller-Jensen-Welch [82], is the following:

$$(*) \qquad \text{If } O^{\#} \notin L[R],\ R \subseteq \omega \text{ then } R \text{ is generic over } L.$$

We must be precise about what is meant by "generic".

Definition. (Stated in Class Theory) A *generic extension* of an inner model M is an inner model $M[G]$ such that for some forcing notion $\mathcal{P} \subseteq M$:

(a) $\langle M, \mathcal{P}\rangle$ is amenable and $\Vdash_p$ is $\langle M, \mathcal{P}\rangle$-definable for $\underset{\sim}{\Delta}_0$ sentences.

(b) $G \subseteq \mathcal{P}$ is compatible, closed upwards and intersects every $\langle M, \mathcal{P}\rangle$-definable dense $D \subseteq \mathcal{P}$.

A set x is *generic* over M if it is an element of a generic extension of M. And x is *strictly generic* over M if $M[x]$ is a generic extension of M.

Though the above definition quantifies over classes, in the special case where $M = L$ and $O^{\#}$ exists these notions are in fact first-order, as all L-amenable classes are $\underset{\sim}{\Delta}_1$ definable over $L[O^{\#}]$. From now on assume that $O^{\#}$ exists.

Theorem A. *The Genericity Conjecture is false.*

The proof is based upon the fact that every real generic over L obeys a certain definability property, expressed as follows.

Fact. If R is generic over L then for some L-amenable class A, $\text{Sat}\langle L, A\rangle$ is *not* definable over $\langle L[R], A\rangle$, where $\text{Sat}\langle L, A\rangle$ is the canonical satisfaction predicate for $\langle L, A\rangle$.

Thus Theorem A is established by producing a real R s.t. $O^{\#} \notin L[R]$ yet $\text{Sat}\langle L, A\rangle$ *is* definable over $\langle L[R], A\rangle$ for each L-amenable A.

A weaker version of the Genericity Conjecture would state: If $O^{\#} \notin L[R]$ then either $R \in L$ or R is generic over some inner model M not containing R. This version of the conjecture is still open. However, this question can also be studied in contexts where $O^{\#}$ does *not* exist, for example when the universe has ordinal height equal to that of the

*Research Supported by the National Science Foundation, Grant #8903380–DMS.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

minimal transitive model of ZF. In the latter context, Mack Stanley [93] has demonstrated the consistency of the existence of a non-constructible real which belongs to every inner model over which it is generic.

Section A A Non-Generic Real below $\mathbf{O^{\#}}$.

We first prove the Fact stated in the introduction.

Lemma 1. *Suppose $R \subseteq \omega$ is generic over L. Then for some L-amenable class A, $Sat\langle L, A\rangle$ is* **not** *definable over $\langle L[R], A\rangle$ with parameters.*

Proof. Let $R \in L[G]$ where $G \subseteq \mathcal{P}$ is generic for $\langle L, \mathcal{P}\rangle$-definable dense classes and $\mathcal{P}$ is L-amenable as in (a), (b) of the definition of generic extension. Let $A = \mathcal{P}$ and suppose that $\mathrm{Sat}\langle L, \mathcal{P}\rangle$ were definable over $\langle L[R], \mathcal{P}\rangle$ with parameters. But the Truth Lemma holds for $G, \mathcal{P}$ for formulas mentioning $G, \mathcal{P}$: $\langle L[G], G, \mathcal{P}\rangle \vDash \phi(G, \mathcal{P})$ iff $\exists p \in G(p \Vdash \phi(G, \mathcal{P}))$, using the fact that $\Vdash$ in $\mathcal{P}$ for $\underset{\sim}{\Delta_0}$ sentences is definable over $\langle L, \mathcal{P}\rangle$ and the genericity of G. So $\mathrm{Sat}\langle L[G], G, \mathcal{P}\rangle$ is definable over $\langle L[G], G, \mathrm{Sat}\langle L, \mathcal{P}\rangle\rangle$, since $\Vdash$ is definable over $\langle L, \mathrm{Sat}\langle L, \mathcal{P}\rangle\rangle$ for arbitrary first-order sentences. Since $\mathrm{Sat}\langle L, \mathcal{P}\rangle$ is definable over $\langle L[G], G, \mathcal{P}\rangle$ we get the definability of satisfaction for the latter structure over itself. This contradicts a well-known result of Tarski. $\dashv$

The rest of this section is devoted to the construction of a real R such that R preserves L-cofinalities ($\mathrm{cof}(\alpha)$ in $L = \mathrm{cof}(\alpha)$ in $L[R]$ for every α) and for every L-amenable A, $\mathrm{Sat}\langle L, A\rangle$ is definable over $\langle L[R], A\rangle$. (The proof has little to do with the Sat operator; any operator from L-amenable classes to L-amenable classes that is "reasonable" is codable by a real. We discuss this further at the end of this section.)

R will generically code a class f which is generic for a forcing of size ∞^+ = least "L-cardinal" greater than ∞. Since this sounds like nonsense we suggest that the reader think of ∞ as some uncountable cardinal of V and then ∞^+ denotes $(\infty^+)^L$. Thus we will define a constructible set forcing $\mathcal{P}^\infty \subseteq L_{\infty^+}$ for adding a generic $f^\infty \subseteq \infty$ such that if $A \subseteq \infty$ is constructible then $\mathrm{Sat}\langle L_\infty, A\rangle$ is definable over $\langle L_\infty[f^\infty], f^\infty, A\rangle$. Then we show how to choose the f^∞'s to "fit together" into an $f \subseteq ORD$ such that $\mathrm{Sat}\langle L, A\rangle$ is definable over $\langle L[f], f, A\rangle$ for each L-amenable A. Finally, we code f by a real R (using the fact that I = Silver Indiscernibles are indiscernibles for $\langle L[f], f\rangle$).

A condition in $\mathcal{P}^\infty$ is defined as follows. Work in L. An *Easton set of ordinals* is a set of ordinals X such that $X \cap \kappa$ is bounded in κ for every regular $\kappa > \omega$. For any $\alpha \in ORD$, 2^α denotes all $f : \alpha \longrightarrow 2$ and $2^{<\alpha} = \cup\{2^\beta | \beta < \alpha\}$. An *Easton set of strings* is a set $D \subseteq \cup\{2^\alpha | \alpha \in ORD\}$ such that $D \cap 2^{<\kappa}$ has cardinality less than κ for every regular $\kappa > \omega$. For any $X \subseteq ORD$ let $\text{Seq}(X) = \cup\{2^\alpha | \alpha \in X\}$. A *condition* in $\mathcal{P}^\infty$ is (X, F, D, f) where:

(a) $X \subseteq \infty$ is an Easton set of ordinals

(b) $F : X \longrightarrow \mathcal{P}(2^\infty) =$ Power Set of 2^∞ such that for $\alpha \in X$, $F(\alpha)$ has cardinality $\leq \alpha$

(c) $D \subseteq \text{Seq}(X)$ is an Easton set of strings

(d) $f : D \longrightarrow \infty$ such that $f(s) >$ length (s) for $s \in D$.

We define extension of conditions as follows. $(Y, G, E, g) \leq (X, F, D, f)$ iff

(i) $Y \supseteq X$, $E \supseteq D$, $G(\alpha) \supseteq F(\alpha)$ for $\alpha \in X$, g extends f

(ii) If $s \in E - D$ then the interval (length $(s) + 1, g(s)]$ contains no element of X, and if $s \subseteq S \in F(\alpha)$ for some $\alpha \leq$ length (s), $\alpha \in X$ then $g(s) \notin C_S$.

We must define C_S. For $S \in 2^\infty$ let $\mu(S) =$ least p.r. closed $\mu > \infty$ such that $S \in L_\mu$ and then $C_S = \{\alpha < \infty | \alpha = \infty \cap$ Skolem hull (α) in $L_{\mu(S)}\}$. Thus C_S is CUB in ∞ and $\langle L_\alpha, S \restriction \alpha \rangle \prec \langle L_\infty, S \rangle$ for sufficiently large $\alpha \in C_S$ (as $S \in$ Skolem hull (α) in $L_{\mu(S)}$ for sufficiently large $\alpha < \infty$). Also note that $T \notin L_{\mu(S)} \longrightarrow C_T \subseteq \text{Lim}\, C_S \cup \alpha$ for some $\alpha < \infty$.

Our goal with this forcing is to produce a generic function f_G from $2^{<\infty}$ into ∞ such that for each $S \subseteq \infty$, $\{f(S \restriction \alpha) | \alpha < \infty\}$ is a good approximation to the complement of C_S. $S \in F(\alpha)$ is a committment that for $\beta > \alpha$, $f(S \restriction \beta) \notin C_S$ (in stronger conditions).

Lemma 2. *If $p \in \mathcal{P}^\infty$ and $\alpha < \infty$, $S \in 2^\infty$, $s \in 2^{<\infty}$ then p has an extension (X, F, D, f) such that $\alpha \in X$, $S \in F(\alpha)$ and $s \in D$.*

Proof. Easy, given the fact that if s needs to be added then we can safely put $f(s) =$ length$(s) + 1$. ⊣

Lemma 3. *$\mathcal{P}^\infty$ has the ∞^+-chain condition (antichains have size $\leq \infty$, all in L of course).*

Proof. Any two conditions $(X, F, D, f), (X, G, D, f)$ are compatible, so an antichain has cardinality at most the number of (X, D, f)'s, which is ∞. ⊣

Lemma 4. *Let G be $\mathcal{P}^\infty$-generic and write f_G for $\cup\{f | (X, F, D, f) \in G$ for some $X, F, D\}$. If $S \in 2^\infty$ then $f_G(S \restriction \alpha) \notin C_S$ for sufficiently large $\alpha < \infty$.*

Proof. G contains a condition (X, F, D, f) such that $0 \in X$ and $S \in F(0)$. If $s \subseteq S, s \notin D$ then $f_G(s) \notin C_S$, by (ii) in the definition of extension. And $S \restriction \alpha \notin D$ for sufficiently large $\alpha < \infty$. ⊣

Lemma 5. *Let G, f_G be as in Lemma 4. If $\alpha < \infty$ is regular, $S \in 2^\infty$, and $\alpha \notin Lim\ C_S$ then $\{f_G(S|\beta)|\beta < \alpha\}$ intersects every constructible unbounded subset of α.*

Proof. Let $A \subseteq \alpha$ be constructible and unbounded in α. We show that a condition (X, F, D, f) can be extended to $(X \cup \{\delta\}, F^*, D \cup \{S \restriction \delta\}, f^*)$ for some δ, where $f^*(S \restriction \delta) \in A$. Choose $\delta < \alpha$ large enough so that $S \restriction \delta$ is not an initial segment of any $T \in \cup\{F(\beta)|\beta \in X \cap \alpha\} - \{S\}$. This is possible since $X \cap \alpha$ is bounded in α and $F(\beta)$ has cardinality $< \alpha$ for each $\beta \in X \cap \alpha$. Then let $f^* = f \cup \{\langle S \restriction \delta, \beta\rangle\}$ where $\beta \in A - C_S - \delta$ and $F^* = F \cup \{\langle \delta, \emptyset\rangle\}$. ⊣

Lemma 6. *$\mathcal{P}^\infty$ preserves cofinalities (i.e., $\mathcal{P}^\infty \Vdash \mathrm{cof}(\alpha) = \mathrm{cof}(\alpha)$ in L for every ordinal α).*

Proof. For regular $\kappa < \infty$ and $p \in \mathcal{P}^\infty$ let $(p)^\kappa$ = "part of p below κ", $(p)_\kappa$ = "part of p at or above κ" be defined in the natural way: if $p = (X, F, D, f)$ then

$$(p)^\kappa = (X \cap \kappa, F \restriction X \cap \kappa, D \cap \mathrm{Seq}\,\kappa, f \restriction D \cap \mathrm{Seq}\,\kappa) \text{ and}$$

$$(p)_\kappa = (X - \kappa, F \restriction X - \kappa, D \cap \mathrm{Seq}(\infty - \kappa), f \restriction D \cap \mathrm{Seq}(\infty - \kappa)).$$

Given p and predense $\langle \Delta_i | i < \kappa\rangle$ we find $q \leq p$ and $\langle \overline{\Delta}_i | i < \kappa\rangle$ such that $\overline{\Delta}_i \subseteq \Delta_i$ for all $i < \kappa$, card $\overline{\Delta}_i \leq \kappa$ for all $i < \kappa$ and each $\overline{\Delta}_i$ is predense below q. (Δ is *predense* if $\{r | r \leq \text{some } d \in \Delta\}$ is dense; it is *predense below* q if every extension of q can be extended into the afore-mentioned set.) This implies that if $\mathrm{cof}(\alpha) \leq \kappa$ in some generic extension $L[G], G$ $\mathcal{P}^\infty$-generic over L, then $\mathrm{cof}(\alpha) \leq \kappa$ in L. Since $\mathcal{P}^\infty$ is ∞^+-CC, this means that $\mathcal{P}^\infty$ preserves all cofinalities.

Given p and $\langle \Delta_i | i < \kappa\rangle$ as above first extend p to $p_0 = (X_0, F_0, D_0, f_0)$ so that $\kappa \in X_0$. Now note that if $r \leq p_0$ then $f^r(s) < \kappa$ for all $s \in D^r - D_0$ of length $< \kappa$ (where $r = (X^r, F^r, D^r, f^r)$), by condition (ii) in the definition of extension. Thus $\mathcal{F} = \{(X^r \cap \kappa, D^r \cap \mathrm{Seq}\,\kappa, f^r \restriction D^r \cap \mathrm{Seq}\,\kappa) | r \leq p_0\}$ is a set of cardinality κ. Let $\langle (\Delta_i^*, (X^i, D^i, f^i)) | i < \kappa\rangle$ be an enumeration in length κ of all pairs from $\{\Delta_i | i < \kappa\} \times \mathcal{F}$.

Now we extend p_0 successively to $p_1 \geq p_2 \geq \ldots$ in κ steps so that $(p_i)^\kappa = (p_0)^\kappa$ for all $i < \kappa$, according to the following prescription: If p_i has been defined, see if it has an extension r_i extending some $d_i \in \Delta_i^*$ such that $(X^{r_i} \cap \kappa, D^{r_i} \cap \mathrm{Seq}\,\kappa, f^{r_i} \restriction D^{r_i} \cap \mathrm{Seq}\,\kappa) = (X^i, D^i, f^i)$. If not then $p_{i+1} = p_i$. If so, select such an r_i, d_i and define p_{i+1} by requiring $(p_{i+1})^\kappa = (p_0)^\kappa, (p_{i+1})_\kappa = (r_i)_\kappa$ except enlarge $F^{p_{i+1}}(\kappa)$ so as to contain $F^{r_i}(\alpha)$ for $\alpha \in X^{r_i} \cap \kappa$. For limit $\lambda \leq \kappa$ let p_λ be the greatest lower bound to $\langle p_i | i < \lambda\rangle$. Finally let $q = p_\kappa$.

Let $\overline{\Delta}_j \subseteq \Delta_j$ consist of all d_i in the above construction that belong to Δ_j, for $j < \kappa$. The claim we must establish is that each $\overline{\Delta}_j$ is predense below q. Here's the proof: suppose $\bar{q} \leq q$ and let $r \leq \bar{q}, r$ extending some element of Δ_j. Choose $i < \kappa$ so that $(\Delta_i^*, (X^i, D^i, f^i)) = (\Delta_j, (X^r \cap \kappa, D^r \cap \operatorname{Seq}\kappa, f^r \restriction D^r \cap \operatorname{Seq}\kappa))$. Clearly at stage $i+1$, it was possible to find r_i, d_i as searched for in the construction. It suffices to argue that $r_i, \bar{q}$ are compatible. Now $(r_i)_\kappa$ is extended by $(p_{i+1})_\kappa$ and hence by $(r)_\kappa$. And $(r_i)^\kappa$ is extended by $(r)^\kappa$, except possibly that $F^{r_i}(\alpha)$ may fail to be a subset of $F^r(\alpha)$ for $\alpha \in X^r \cap \kappa$. And note that the extension $(r_i)_\kappa \geq (r)_\kappa$ obeys all restraint imposed by $F^{r_i}(\alpha)$ for $\alpha \in X^r \cap \kappa$ since we included $F^{r_i}(\alpha)$ in $F^{p_{i+1}}(\kappa)$. Thus r_i and $\bar{q}$ are both extended by r, provided we only enlarge $F^r(\alpha)$ for $\alpha \in X^r \cap \kappa$ to include $F^{r_i}(\alpha)$. $\dashv$

For future reference we state:

Corollary 6.1. *Suppose $\kappa < \infty$ is regular and $\Delta \subseteq \mathcal{P}^\infty$ is predense. Let $\mathcal{P}^\infty_\kappa = \{(p)_\kappa | p \in \mathcal{P}^\infty\}, \mathcal{P}^{\infty;\kappa} = \{p \in \mathcal{P}^\infty | X^p \subseteq \kappa$ and Range $(f^p) \subseteq \kappa\}$ with the notion $\leq$ of extension defined as for $\mathcal{P}^\infty$. Then for any $q \in \mathcal{P}^\infty_\kappa$ there is $q' \leq q$ such that $\Delta^{q'} = \{r \in \mathcal{P}^{\infty;\kappa} | r \cup q'$ meets $\Delta, F^r(\alpha) \subseteq F^{q'}(\kappa)$ for all $\alpha \in X^r\}$ is predense on $\mathcal{P}^{\infty;\kappa}$.*

Proof. As in the proof of Lemma 6, successively extend q (after guaranteeing $\kappa \in X^q$) in κ steps to q' so that for any (X, D, f) if $r \cup q''$ meets Δ for some $q'' \leq q'$, some r such that $(X^r, D^r, f^r) = (X, D, f)$ then $r \cup q'$ meets Δ for some such r, where $F^r(\alpha) \subseteq F^{q'}(\kappa)$ for all $\alpha \in X^r$. Now note that if $r_0 \in \mathcal{P}^{\infty;\kappa}$ then $r_0 \cup q'$ has an extension meeting Δ so there is r_1 such that $(X^{r_1}, D^{r_1}, f^{r_1}) = (X^{r_0}, D^{r_0}, f^{r_0})$ and $r_1 \in \Delta^{q'}$. But then r_0 is compatible with r_1 so $\Delta^{q'}$ is predense on $\mathcal{P}^{\infty;\kappa}$, as desired. $\dashv$

Corollary 6.2. *$\mathcal{P}^\infty \Vdash GCH$.*

Proof. Suppose $f^\infty : \operatorname{Seq}(\infty) \longrightarrow \infty$ is $\mathcal{P}^\infty$-generic. It suffices to show that if $\kappa \leq \infty$ is regular, $A \subseteq \kappa$, $A \in L[f^\infty]$ then $A \in L[f^\infty \restriction \operatorname{Seq}(\kappa)]$. But the proof of Lemma 6 shows that given any $p \Vdash \dot{A} \subseteq \kappa$ there is $q \leq p$ such that for any $i < \kappa$, $\{r \leq q | (r)_\kappa = (q)_\kappa$ and r decides "$i \in \dot{A}$"$\}$ is predense below q. This proves that there is $q \leq p$ such that $q \Vdash \dot{A} \in L[\dot{f}^\infty \restriction \operatorname{Seq}(\kappa)]$ and so by the genericity of f^∞, $A \in L[f^\infty \restriction \operatorname{Seq}(\kappa)]$. $\dashv$

Next we embark on a series of lemmas aimed at showing that $\mathcal{P}^\infty$-generics actually exist in $L[O^\#]$ when ∞ is any Silver indiscernible.

Lemma 7. *Suppose $i < j$ are adjacent countable Silver indiscernibles. Let $\pi = \pi_{ij}$ denote the elementary embedding $L \longrightarrow L$ which shifts each of the Silver indiscernibles $\geq i$ to the* *next one and leaves all other Silver indiscernibles fixed. Then there is a $\mathcal{P}_i^j$-generic G_i^j such that if (X, F, D, f) belongs to G_i^j and $S \subseteq i$, $S \in L$ then $f(\pi(S) \restriction \alpha) \notin C_{\pi(S)}$ for all $\pi(S) \restriction \alpha \in D$.*

Proof. For any $k \in \omega$ let $\ell_1 < \cdots < \ell_k$ be the first k Silver indiscernibles greater than j and let $j_k = j^+ \cap \Sigma_1$ Skolem hull of $j + 1 \cup \{\ell_1 \ldots \ell_k\}$ in L, $i_k = i^+ \cap \Sigma_1$ Skolem hull of $i + 1 \cup \{\ell_1 \ldots \ell_k\}$ in L. (Of course i^+, j^+ denote the cardinal successors to i, j *in* L.) Let $j_k^* =$ least p.r. closed ordinal $\alpha > j_k$ such that $L_\alpha \vDash j$ is the largest cardinal. Finally let $C_k = \{\gamma < j | \gamma = j \cap \Sigma_1$ Skolem hull $(\gamma \cup \{j\} \cup \{\ell_1 \ldots \ell_k\})$ in $L\}$, a CUB subset of j.

Now note that if $S \subseteq i, S \in L - L_{i_k}$ then $C_{\pi(S)} \subseteq C_k \cup \gamma$ for some $\gamma < i$. For, $\mu_{\pi(S)}$ is greater than or equal to j_k^* since otherwise $\pi(S)$ belongs to L_{j_k} and hence S belongs to L_{i_k}. Thus $C_{\pi(S)} \subseteq C_k \cup \gamma$ for some $\gamma < j$ since C_k is an element of $L_{j_k^*}$. But the least such γ is definable from elements of $i\cup$ (Silver Indiscernibles $\geq j$), so must be less than i.

Also note that the L-cofinality of j_k is equal to j : Consider $M =$transitive collapse of Σ_1 Skolem hull of $j + 1 \cup \{\ell_1 \ldots \ell_k\}$. There is a partial $\underline{\Sigma_1}(M)$ function from a subset of j onto j_k, all of whose restrictions to ordinals $\gamma < j$ have range bounded in j_k. (This is why we are using Σ_1 Skolem hulls rather than full Σ_ω Skolem hulls.) Thus the L-cofinalities of j_k and j are the same, namely j.

Thus we may conclude the following: The set $\{\pi(S) | S \subseteq i, S \in L_{i_k}\} \in L_{j_k}$ (since it is a constructible bounded subset of L_{j_k}) and if $S \subseteq i$, $S \in L - L_{i_k}$ then $C_{\pi(S)}$ is disjoint from (i, γ_k), where $\gamma_k =$ least element of C_k greater than i.

Now we see how to build G_i^j. We describe an ω-sequence $p_0 \geq p_1 \geq \ldots$ of conditions in $\mathcal{P}_i^j$ and take $G_i^j = \{p \in \mathcal{P}_i^j | p_k \leq p$ for some $k\}$. Let $\langle \Delta_k | k \in \omega \rangle$ be a list of all constructible dense sets on $\mathcal{P}_i^j$ so that for all k, Δ_k belongs to the Σ_1 Skolem hull in L of $i \cup \{i, j, \ell_1 \ldots \ell_{k+1}\}$. This is possible since any constructible dense set on $\mathcal{P}_i^j$ belongs to $L_{j^{++}}$ and hence to the Σ_1 Skolem hull in L of $i \cup \{i, j, \ell_1 \ldots \ell_k\}$ for some k. We inductively define $p_0 \geq p_1 \geq \ldots$ so that p_k belongs to the Σ_1 Skolem hull in L of $i^+ \cup \{j, \ell_1 \ldots \ell_k\}$. Let p_0 be the weakest condition in $\mathcal{P}_i^j$; $p_0 = (\emptyset, \emptyset, \emptyset, \emptyset)$. Suppose that $k > 0$ and p_{k-1} has been defined. Write $p_{k-1} = (X, F, D, f)$. First obtain $\bar{p}_k$ by adding i to X if necessary and defining or enlarging $F(i)$ so as to include $\{\pi(S) | S \subseteq i, S \in L_{i_k}\}$. Then choose $p_k \leq \bar{p}_k$ to be L-least so that p_k meets Δ_{k-1}. This completes the construction.

We show that $p_k \in \Sigma_1$ Skolem hull in L of $i^+ \cup \{j, \ell_1 \ldots \ell_k\}$. By induction p_{k-1} belongs to this hull and by choice of $\langle \Delta_k | k \in \omega \rangle$, so does Δ_{k-1}. Now $\{\pi(S) | S \subseteq i, S \in L_{i_k}\}$ is the range of $f \restriction i$ where f is a $\underline{\Sigma_1}(L)$ partial function with parameters $j, \ell_1 \ldots \ell_k$. The latter is because Range$(\pi \restriction i_k)$ is just $j_k \cap \Sigma_1$ Skolem hull in L of $i \cup \{j, \ell_1 \ldots \ell_k\}$. But given a parameter x for the domain of this $\underline{\Sigma_1}(L)$ partial function, its range becomes Σ_1-definable in the sense that it is in the Σ_1 Skolem hull in L of $\{x, j, \ell_1 \dots \ell_k\}$. As x can be chosen equivalently as an ordinal $< i^+$, we get that $\{\pi(S) | S \subseteq i, S \in L_{i_k}\}$ belongs to the Σ_1 Skolem hull in L of $i^+ \cup \{j, \ell_1 \dots \ell_k\}$. Thus so does p_k. (Actually x can be chosen to be i_k.)

Finally we must check that if $p_k = (X_k, F_k, D_k, f_k)$ then $f_k(\pi(S) \restriction \alpha) \notin C_{\pi(S)}$ for all $\pi(S) \restriction \alpha \in D_k$, all $S \subseteq i$ in L. Assume that this is true for smaller k and we check it for k. Now if $S \in L_{i_k}$ then this is guaranteed by the fact that $\pi(S) \in \overline{F}_k(i)$, where $\bar{p}_k = (\overline{X}_k, \overline{F}_k, D_{k-1}, f_{k-1})$. If $S \in L - L_{i_k}$ then $C_{\pi(S)}$ is disjoint from (i, γ_k), where $\gamma_k = j \cap \Sigma_1$ Skolem hull in L of $\gamma_k \cup \{j\} \cup \{\ell_1 \dots \ell_k\}$ and $\gamma_k > i$. But then $\gamma_k > i^+$ so $C_{\pi(S)}$ is disjoint from $(i, \bar{\gamma}_k)$ where $\bar{\gamma}_k = \sup(j \cap \Sigma_1$ Skolem hull in L of $i^+ \cup \{j\} \cup \{\ell_1 \dots \ell_k\})$. Since $p_k \in \Sigma_1$ Skolem hull in L of $i^+ \cup \{j\} \cup \{\ell_1 \dots \ell_k\}$, it follows that $\text{Range}(f_k) \subseteq \bar{\gamma}_k$ and hence $\text{Range}(f_k)$ is disjoint from $C_{\pi(S)}$. $\dashv$

Lemma 8. *Suppose $i < j$ are adjacent Silver indiscernibles, G_i^j is $\mathcal{P}_i^j$-generic over L as in Lemma 7 and G^i is $\mathcal{P}^i$-generic over L. Then there exists G^j which is $\mathcal{P}^j$-generic over L such that $G_i^j = \{(p)_i | p \in G^j\}$ and $q \in G^i \longleftrightarrow \pi_{ij}(q) \in G^j$.*

Proof. As before, let $\mathcal{P}^{j,i} \subseteq \mathcal{P}^j$ consist of all $p = (X^p, F^p, D^p, f^p)$ in $\mathcal{P}^j$ such that $X^p \subseteq i$ and Range $(f^p) \subseteq i$. For any $p \in \mathcal{P}^{j,i}$ we modify p to $\bar{p}$ as follows. For $S \in F^p(\alpha)$, $i \in C_S$ let $\bar{S} = \pi_{ij}(S \restriction i)$. For $S \in F^p(\alpha), i \notin C_S$ let $T \subseteq i$ be L-least so that (T, C_T), (S, C_S) agree through $\sup(C_S \cap i)$ and let $\overline{S} = \pi_{ij}(T)$. Then $F^{\bar{p}}(\alpha)$ consists of all $\overline{S}$ for $S \in F^p(\alpha)$. Otherwise $p, \bar{p}$ agree: $(X^p, D^p, f^p) = (X^{\bar{p}}, D^{\bar{p}}, f^{\bar{p}})$.

If $p \in \mathcal{P}_i^j$ and $i \in X^p$ we let $Q(p)$ denote $\{q \in \mathcal{P}^{j,i} | F^q(\alpha) \subseteq F^p(i)$ for all $\alpha \in X^q.\}$ Now define $\overline{G}^j = \{p \in \mathcal{P}^j | (p)_i \in G_i^j, i \in X^p, (p)^i \in Q((p)_i)$ and $\overline{(p)^i} \in \pi_{ij}[G^i]\}$. Note that if p_0, p_1 belong to $\overline{G}^j$ then p_0, p_1 are compatible because $(p_0)_i, (p_1)_i$ are compatible, the restraints from $(p_0)^i, (p_1)^i$ are "covered" by $F^{p_0}(i), F^{p_1}(i)$ and $\overline{(p_0)^i}, \overline{(p_1)^i}$ impose at least as much restraint below i as do $(p_0)^i, (p_1)^i$. Note that if $G^j = \{p | \bar{p} \leq p$ for some $\bar{p} \in \bar{G}^j\}$ then G^j is compatible, closed upwards and $G_i^j = \{(p)_i | p \in G^j\}$. Also $q \in G^i \longleftrightarrow \pi_{ij}(q) \in G^j$, using the hypothesis that G_i^j satisfies Lemma 7. So it only remains to show that $\overline{G}^j$ meets all constructible predense $\Delta \subseteq \mathcal{P}^j$.

The first Corollary to Lemma 6 states that it is enough to show that $\overline{G}_i^j = \{(p)_i | p \in \overline{G}^j\}$ meets all constructible predense $\Delta \subseteq \mathcal{P}_i^j$ and that for $p \in \overline{G}_i^j, \{q \in Q(p) | q = (r)^i$ for some $r \in \overline{G}^j\}$ meets all constructible $\Delta \subseteq Q(p)$ which are predense on $\cup\{Q(p^*) | p^* \leq p\} = \mathcal{P}^{j,i}$. The former assertion is clear by the $\mathcal{P}_i^j$-genericity over L of $G_i^j = \overline{G}_i^j$. To prove the latter assertion we must show that for $p \in \overline{G}^j_i$, $\{q \in Q(p) | \bar{q} \in \pi_{ij}[G^i]\}$ meets every constructible $\Delta \subseteq Q(p)$ which is predense on $\mathcal{P}^{j,i}$. Given such a Δ, let $\overline{\Delta} \subseteq \mathcal{P}^i$ be defined by $\overline{\Delta} = \{r \in \mathcal{P}^i | \pi_{ij}(r) = \bar{q}$ for some q meeting $\Delta\}$. Note that $\overline{\Delta}$ is constructible because it equals $\{r \in \mathcal{P}^i | r = \pi_{ij}^{-1}(\bar{q})$ for some q meeting $\Delta\}$ and Δ has L-cardinality $\leq i$. We claim that $\overline{\Delta} \subseteq \mathcal{P}^i$ is predense on $\mathcal{P}^i$. Indeed, if $r \in \mathcal{P}^i$ then $\pi_{ij}(r) \in \mathcal{P}^{j,i}$ and therefore can be extended to some q meeting Δ. As $\bar{q} = \pi_{ij}(t)$ for some $t \leq r$ we have shown that r can be extended into $\overline{\Delta}$. By the $\mathcal{P}^i$-genericity of G^i, choose $r \in \overline{\Delta} \cap G^i$. Then $\pi_{ij}(r) = \bar{q}$ where q meets Δ; clearly $\bar{q} \in \pi_{ij}[G^i]$. ⊣

Lemma 9. *Let $i_1 < i_2 < \ldots$ denote the first ω-many Silver indiscernibles and i_ω their supremum. Then there exist $\langle G^{i_n} | n \geq 1\rangle$ such that G^{i_n} is $\mathcal{P}^{i_n}$-generic over L and whenever $\pi : L \longrightarrow L$ is elementary, $\pi(i_\omega) = i_\omega$ we have $p \in G^{i_n} \longleftrightarrow \pi(p) \in G^{\pi(i_n)}$.*

Proof. Note that any π as in the statement of the lemma restricts to an increasing map from $\{i_n | n \geq 1\}$ to itself, so $G^{\pi(i_n)}$ makes sense. We define G^{i_n} by induction on $n \geq 1$. Select G^{i_1} to be the $L[O^\#]$-least $\mathcal{P}^{i_1}$-generic (over L). Select $G^{i_2}_{i_1}$ as in Lemma 7 and use Lemma 8 to define G^{i_2} from $G^{i_2}_{i_1}, G^{i_1}$. Now suppose that G^{i_n} has defined, $n \geq 2$. Then define $G^{i_{n+1}}_{i_n}$ to be $\{p \in \mathcal{P}^{i_{n+1}}_{i_n} | \pi_{i_1 i_n}(q) \leq p$ for some $q \in G^{i_2}_{i_1}\}$ where $\pi_{i_1 i_n}(i_m) = i_{m+n-1}$ for $m < \omega, \pi_{i_1 i_n}(j) = j$ for $j \in I - i_\omega$. Then $G^{i_{n+1}}_{i_n}$ is $\mathcal{P}^{i_{n+1}}_{i_n}$-generic, using the $\leq i_1$-closure of $\mathcal{P}^{i_2}_{i_1}$ and the fact that the collection of constructible dense subsets of $\mathcal{P}^{i_{n+1}}_{i_n}$ is the countable union of sets of the form $\pi_{i_1 i_n}(A), A$ of L-cardinality i_1. Moreover $G^{i_{n+1}}_{i_n}$ obeys the condition of Lemma 7 since $G^{i_2}_{i_1}$ does and $\pi_{i_1 i_n}$ is elementary. Now define $G^{i_{n+1}}$ from $G^{i_{n+1}}_{i_n}, G^{i_n}$ using Lemma 8.

To verify $p \in G^{i_n} \longleftrightarrow \pi(p) \in G^{\pi(i_n)}$, note that this depends only on $\pi \restriction L_{i_\ell}$ for some $\ell < \omega$ and any such map is the finite composition of maps of the form π_m, where $\pi_m(i_n) = i_{n+1}$ for $n \geq m$, $\pi_m(i_n) = i_n$ for $1 \leq n < m$. So we need only verify that for each $m, n, p \in G^{i_n} \longleftrightarrow \pi_m(p) \in G^{\pi_m(i_n)}$. This is trivial unless $m \leq n$ as $m > n \longrightarrow \pi_m(p) = p$ for $p \in G^{i_n} = G^{\pi_m(i_n)}$. Finally we prove the statement by induction on $n \geq m$. If $n = m$ then it follows from the fact that $G^{i_{n+1}}$ was defined from $G^{i_{n+1}}_{i_n}, G^{i_n}$ so as to obey the conclusion of Lemma 8. Suppose it holds for $n \geq m$ and we wish to demonstrate the property for $n+1$. But $G^{i_{n+1}}$ is defined from $G^{i_{n+1}}_{i_n}, G^{i_n}$ as $G^{i_{n+2}}$ is defined from $G^{i_{n+2}}_{i_{n+1}}, G^{i_{n+1}}$. Clearly $\pi_m[G^{i_{n+1}}_{i_n}] \subseteq G^{i_{n+2}}_{i_{n+1}}$ and by induction $\pi_m[G^{i_n}] \subseteq G^{i_{n+1}}$. Thus $p \in G^{i_{n+1}} \longrightarrow \pi_m(p) \in G^{\pi_m(i_{n+1})}$. Conversely, $p \notin G^{i_{n+1}} \longrightarrow p$ incompatible with some $q \in G^{i_{n+1}} \longrightarrow \pi_m(p)$ incompatible with some $\pi_m(q) \in G^{\pi_m(i_{n+1})} \longrightarrow \pi_m(p) \notin G^{\pi_m(i_{n+1})}$. ⊣

Lemma 10. *There exist $\langle G^i | i \in I\rangle$ such that G^i is $\mathcal{P}^i$-generic over L and whenever $\pi : L \longrightarrow L$ is elementary, $p \in G^i \longleftrightarrow \pi(p) \in G^{\pi(i)}$.*

Proof. Let t denote a Skolem term for L; thus $L = \{t(j_1 \dots j_n) | t$ a Skolem term, t n-ary, $j_1 < \dots < j_n$ in $I\}$. Now define $t(j_1 \dots j_n) \in G^i$ iff $t(\sigma(j_1) \dots \sigma(j_n)) \in G^{\sigma(i)}$ where σ is the unique order-preserving map from $\{i, j_1 \dots j_n\}$ onto an initial segment of I. (G^i for $i < i_w$ is defined in Lemma 9.) We verify that this is well-defined: if $t_1(j_1 \dots j_n) = t_2(k_1 \dots k_m)$ then let σ^* be the unique order-preserving map from $\{i, j_1 \dots j_n, k_1 \dots k_m\}$ onto an initial segment of I. Then $t_1(\sigma^*(j_1) \dots \sigma^*(j_n)) = t_2(\sigma^*(k_1) \dots \sigma^*(k_m))$. But $t_1(\sigma^*(j_1) \dots \sigma^*(j_n)) \in G^{\sigma^*(i)}$ iff $t_1(\sigma_1(j_1) \dots \sigma_1(j_n)) \in G^{\sigma_1(i)}$ where σ_1 is the unique order-preserving map from $\{i, j_1 \dots j_n\}$ onto an initial segment of I, using Lemma 9. The analogous statement holds for t_2, so our definition is well-defined. The property $p \in G^i \longleftrightarrow \pi(p) \in G^{\pi(i)}$ is clear, using our definition. ⊣

Now we are almost done. For any $i \in I$ let $f^i = \cup\{f^p | p \in G^i\}$. Thus $f^i : 2^{<i} \longrightarrow i$. And let $f = \cup\{f^i | i \in I\}$, so $f : 2^{<\infty} \longrightarrow \infty$ (∞ now denotes *real* ∞, that is, $\infty =$ ORD).

Lemma 11. *(a) For any L-amenable $A \subseteq ORD$, $SAT\langle L, A\rangle$ is definable over $\langle L[f], f, A\rangle$.*

(b) I is a class of indiscernibles for $\langle L[f], f\rangle$.

(c) $L[f] \vDash GCH$.

Proof. (a) We treat A as an L-amenable function $A : \infty \longrightarrow 2$. By Lemmas 4,5 we have that for sufficiently large L-regular $\alpha, \alpha \in Lim\ C_A \longleftrightarrow$ Range of $f \restriction \{A \restriction \beta | \beta < \infty\}$ intersects every constructible unbounded subset of α (where C_A is defined for A to be the limit of $C_{A\restriction i}, i \in I$). But for α sufficiently large in C_A, $\langle L_\alpha, A \restriction \alpha\rangle \prec \langle L, A\rangle$ so Sat$\langle L, A\rangle$ is definable over $\langle L[f], f, A\rangle$.

(b) Clear by Lemma 10.

(c) By Corollary 6.2. ⊣

Finally, using the technique of the proof of Theorem 0.2 of Beller-Jensen-Welch [82], there is a real R such that f is definable over $L[R]$ and $I^R = I$. Thus we conclude.

Theorem 12. *There is a real $R \in L[O^\#]$ such that:*

(a) *$L, L[R]$ have the same cofinalities*

(b) *$I^R = I$*

(c) *If A is an L-amenable class then $Sat\langle L, A\rangle$ is definable over $\langle L[R], A\rangle$.*

By Lemma 1 we conclude:

Theorem A. *The Genericity Conjecture is false.*

We close this section by mentioning a generalization of the above treatment of the SAT operator to other operators on classes. For simplicity we first state our result in terms of ω_1, rather than ∞.

Theorem 13. *Assume that $O^{\#}$ exists. Suppose F is a constructible function from $\mathcal{P}_L(\omega_1)$ to itself, where $\mathcal{P}_L(\omega_1)$ = all constructible subsets of (true) ω_1. Then there exists a real $R <_L O^{\#}$ such that $F(A)$ is definable over $\langle L_{\omega_1}[R], A\rangle$ for all $A \in \mathcal{P}_L(\omega_1)$.*

Proof. Choose $\alpha < \omega_1$ so that F is definable in L from parameters in $\alpha \cup (I - \omega_1)$. Also we may construct F', defined from the same parameters, so that for any $A \in \mathcal{P}_L(\omega_1)$, $F(A)$ is definable over $\langle L_{\omega_1}, A, B\rangle$ for any unbounded $B \subseteq F'(A)$. Finally note that we may assume that $F'(A) \subseteq C_A$ for all A (where A is viewed as an element of 2^{ω_1}) since C_A is definable over $\langle L_{\omega_1}, A, B\rangle$ for any unbounded $B \subseteq C_A$.

For any $i \in I$, $\alpha \leq i \leq \omega_1$, let F'_i be defined in L just like F', but with ω_1 replaced by i. Also define $\mathcal{P}^i$ as before but with C_S replaced by $F'_i(S)$ (viewing $S \in 2^i$ as a subset of i). Then as before we can construct a generic $f : 2^{<\omega_1} \longrightarrow \omega_1$ so that for any $A \in \mathcal{P}_L(\omega_1)$, $F(A)$ is definable over $\langle L_{\omega_1}[f], A\rangle$. Finally code f generically by a real using the fact that α is countable and $I \cap (\alpha, \omega_1)$ is a set of indiscernibles for $\langle L_{\omega_1}[f], f\rangle$. ⊣

To deal with operators on L-amenable classes, we have to keep track of parameters.

Definition. Suppose $i < j$ belong to I and F_i is a counstructible function from $\mathcal{P}_L(i)$ to itself. Then $F_i^j : \mathcal{P}_L(j) \longrightarrow \mathcal{P}_L(j)$ is defined as follows: Write $F_i = t(\alpha, i, \vec{k})$ where t is a Skolem term for L, $\alpha < i$ and $\vec{k}$ are Silver indiscernibles greater than j. Then $F_i^j = t(\alpha, j, \vec{k})$.

Also define F_i^∞ : L-amenable classes $= \mathcal{P}_L(\infty) \longrightarrow \mathcal{P}_L(\infty)$ as follows: Given an L-amenable A choose t and α so that for all $j \in I$ greater than α, $A \cap j = t(\alpha, j, \vec{k})$ where $\vec{k}$ are Silver indiscernibles greater than j. Then $F_i^\infty(A) = \cup\{F_i^j(A \cap j) | \alpha < j \in I\}$. An operator $F : \mathcal{P}_L(\infty) \longrightarrow \mathcal{P}_L(\infty)$ is *countably constructible* if it is of the form $F_{\omega_1}^\infty$ where F_{ω_1} is a constructible function from $\mathcal{P}_L(\omega_1)$ to itself.

Theorem 14. *Assume that $O^{\#}$ exists and $F : \mathcal{P}_L(\infty) \longrightarrow \mathcal{P}_L(\infty)$ is countably constructible. Then there exists $R <_L O^{\#}$ such that $F(A)$ is definable over $\langle L[R], A\rangle$ for all $A \in \mathcal{P}_L(\infty)$.*

Proof. Apply Theorem 13 to F_{ω_1} where $F = F_{\omega_1}^\infty$. The resulting real R satisfies the conclusion of the present Theorem. ⊣

Remarks. (a) The definitions of $F(A)$ over $\langle L_{\omega_1}(R), A\rangle$, $\langle L[R], A\rangle$ in Theorems 13, 14 respectively are independent of A.

(b) If $F : \mathcal{P}_L(\omega_1) \longrightarrow \mathcal{P}_L(\omega_1)$ is constructible then there exists a set-generic extension of L in which there is a real R obeying the conclusion of Theorem 13. However we cannot expect there to be such a real in $L[O^\#]$, or even compatible with the existence of $O^\#$. The key feature of our forcing $\mathcal{P}$ is that not only can it be used to produce a real R obeying the conclusion of Theorem 12 but such a real can be found in $L[O^\#]$. If one is willing to entirely ignore compatibility with $O^\#$ then there are forcings far simpler than ours which achieve the effect of Theorem 14 for any F : classes $\longrightarrow$ classes, over any model of Gödel-Bernays class theory.

References

1. Beller-Jensen-Welch, *Coding the Universe*, Cambridge University Press, 1982.
2. Friedman, *Minimal Universes*, to appear, Advances in Mathematics, 1993.
3. M. Stanley, *A Non-Generic Real Incompatible with* $0^\#$, To appear, 1993.