Hyperfine Structure Theory and Gap 1 Morasses

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Abstract

Using the Friedman-Koepke Hyperfine Structure Theory of [2], we provide a short construction of a gap 1 morass in the constructible universe.

Introduction

The constructible universe L of set theory is defined as the class of sets definable in a transfinite process as follows: Start with an empty L_0, for L_α already defined let $L_{\alpha+1}$ consist of all subsets of L_α definable by $\in$-formulae, and for limit ordinals λ take the union of all previous stages of the construction, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$. Finally $L = \bigcup_{\alpha\in\mathrm{On}} L_\alpha$.

As a consequence of its very concrete definition, L has some fundamental properties which are unprovable in ZFC alone. For example, Gödel defined this model to prove the consistency of ZFC with the continuum hypothesis (CH). His proof is based on a *condensation lemma* which states that Σ_1-substructures of L condense down to stages of L.

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In contrast to the simplicity of its definition, the proofs of some of L's most important properties such as the $\Box$-principle or the covering lemma can be rather complex. Jensen [5] in 1972 established those results, using his *fine structure theory.* Even today, after 30 years of development, Jensen's method remains challenging.

In the early seventies, Silver found a different approach — the Silver machines (see Richardson [7]). These machines reduce set-theoretic properties to calculations with sets of ordinals. In analogy to the L-hierarchy, a hierarchy of algebras M^δ is defined. And analogous to the condensation lemma is the *collapsing property*: closed structures (which are produced by a hull operator) condense down to stages of the machine. In contrast to the L-hierarchy, very little happens in the passage from M^δ to $M^{\delta+1}$. This is guaranteed by a certain *finiteness property* which codes all information needed for this step in a finite set which itself has a simple form. A nice introduction to Silver machines can be found in the third author's Diplom thesis [6].

Another approach is due to Friedman and Koepke [2]; it incorporates the finiteness property and other ideas of Silver machines into the L-hierarchy. The advantage of the resulting *hyperfine structure theory* is that it not only achieves the finiteness property, but also preserves the natural and simple intuitions inherent in the L-hierarchy. In this article we use this theory to build gap 1 morasses. *Morasses* are combinatorial structures invented by Jensen as a tool to construct infinite structures from structures of smaller cardinality, e.g., a structure of size $\aleph_{n+1}$ can be built from countable structures using a gap n morass. An important application is the gap $(n+1)$ transfer theorem (requiring a gap n morass). For a discussion of these ideas, see Devlin [1].

Jensen's definition of a *gap 1 morass* is repeated in definition 10 below. Richardson [7] has a construction of such a morass using Silver machines. In this article we construct a gap 1 morass using a precise language for hyperfine structures together with a property called *type preservation* (lemma 8); this basically says that isomorphic hulls are still isomorphic when mapped in a Σ_1-preserving way; using type preservation one can avoid lengthy calculations using terms in the language of hyperfine structure theory.

The construction we give is of a gap 1 morass at ω_1. It easily generalises to give a gap 1 morass at any regular cardinal.

Gap 2 morasses can also be constructed using the hyperfine structure theory. For this we refer the reader to the forthcoming [3].

Notation

The basic concepts of set theory (especially the constructible universe L) are assumed to be known. Any notation and definition not explained is standard and may, e.g., be found in Jech [4].

We use the usual logical symbols: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\exists$ (exists), $\forall$ (for all), $\to$ (implies), (, and) (parentheses).

For two sets x and y we write $x \cong y$ if x and y are isomorphic (i.e., there exists a 1-1 function from x onto y which preserves all structures on x; the structures will be clear from the context). Furthermore, we write $x \subset y$ if x is a (not necessarily proper) subset of y. For a well-ordering $\langle Z, <_Z \rangle$ and a set $X \subset Z$ let $\operatorname{lub} X$ (least upper bound) be the $<_Z$-least $z \in Z$ s. t. $\forall x \in X\ x < z$. As usual small Greek letters will denote ordinals.

Let $f\colon x \to y$; we write $\operatorname{dom} f$ for the domain and $\operatorname{range} f$ for the range of f. ${}^{<\omega}x$ is the set of all finite sequences in x. If x and y are ordered sets and f is a function which preserves these orders we write $f\colon x \xrightarrow{\text{o. p.}} y$.

The Friedman-Koepke Hyperfine Structure Theory

Let's recall the basic definitions and properties. See [2] for details and proofs. The main tools of the theory are *locations*, also referred to as *names*, and the corresponding *hulls*. Locations are triples of the form $(\alpha, \varphi, \vec{x})$ well-ordered by $\tilde{\leq}$ (such a location will be called an *α-location*, we will also refer to α as the level of this location). For a given location s we write $s = (\alpha(s), \varphi_{n(s)}, \vec{x}(s))$, where a canonical list $\varphi_0, \varphi_1, \cdots$ of formulas has been fixed. The basic operations are:

Interpretation $I(\alpha, \varphi, \vec{x}) = \{y \in L_\alpha \mid L_\alpha \models \varphi(y, \vec{x})\}$

Naming For $y \in L$ let $N(y) = (\alpha, \varphi, \vec{x})$ be $\tilde{\leq}$-least s. t. $I(\alpha, \varphi, \vec{x}) = y$.

Skolem function $S(\alpha, \varphi, \vec{x})$ is the least $y \in L_\alpha$ s. t. $L_\alpha \models \varphi(y, \vec{x})$ if exists; else $S(\alpha, \varphi, \vec{x})\uparrow$ (undefined).

We say that $(\alpha, \varphi, \vec{x}) \in X \subset L$ if α and each component of $\vec{x}$ are elements of X. A set or class $X \subset L$ is *constructibly closed* iff X is closed under applications of I, N, and S. We denote the constructible *closure* or *hull* of X by $L\{X\}$. If X is constructibly closed and $\pi\colon X \cong M$ is the Mostowski collapse, then $M = L_\alpha$ for some $\alpha \in \mathrm{On}$ and the basic operations are preserved by π.

The *fine constructible hierarchy* is given by

$$L_s = \left(L_{\alpha(s)}, \in, <_L, I, N, S\restriction s\right)$$

where $S\restriction s$ means that S is applied to locations in $L_{\alpha(s)}$ and to $\alpha(s)$-locations which are below s with respect to $\tilde{\leq}$ (the latter will for that purpose also be considered elements of the structure, but not of the domain of I). Now the definition of closure extends to structures L_s for a location s, namely a set $X \subset L_{\alpha(s)}$ is closed in L_s ($X \triangleleft L_s$) if it is closed under its operations (S can be applied to top-locations below s if their third component is an element of X). The hull $L_s\{X\}$ is defined similarly. Again, we have *condensation*: There is a unique isomorphism $\pi\colon L_s\{X\} \cong L_{\bar{s}}$ for some $\bar{s}$. Locations are mapped component-wise; if the first component is $\alpha(s)$ it is mapped to $\alpha(\bar{s})$. For notational convenience we write $\pi(s) = \bar{s}$.

For finite sets $p, q \subset L_{\alpha(s)}$ define $p <^* q$ iff $\max_{<_L}(p \bigtriangleup q) \in q$ ($\bigtriangleup$ is the symmetric difference). If a finite set is used as a parameter to a formula, it is taken as a $<_L$-increasing tuple.

Additionally, we have a *finiteness property*, *monotonicity*, *continuity*, and a *compactness property*:

Finiteness Property For an α-location s there exists $z \in L_\alpha$ s. t. for any $X \subset L_\alpha$ we have $L_{s^+}\{X\} \subset L_s\{X \cup \{z\}\}$ where s^+ denotes the immediate successor of s in the well-ordering of locations; $z = S(s)$ is as required.

Monotonicity For α-locations $s \tilde{\leq} t$: $\forall X \subset L_\alpha\ L_s\{X\} \subset L_t\{X\}$
For s, t α-, β-locations respectively, where $\alpha < \beta$:

$$L_s\{X\} \subset L_t\{X \cup \{\alpha\}\}$$

Continuity For locations of the form $s = (\alpha, \varphi_0, \emptyset)$ for $\lim \alpha$ and $X \subset L_\alpha$:

$$L_s\{X\} = L\{X\} = \bigcup_{\beta<\alpha} L_{(\beta,\varphi_0,\emptyset)}\{X \cap L_\beta\}$$

For $s = (\alpha+1, \varphi_0, \emptyset)$ and $X \subset L_\alpha$:

$$L_s\{X \cup \{\alpha\}\} \cap L_\alpha = L\{X \cup \{\alpha\}\} \cap L_\alpha$$
$$= \bigcup \{L_t\{X\} \mid t \text{ an } \alpha\text{-location}\}$$

For $s = (\alpha, \varphi, \vec{x})$ a $\widetilde{\leq}$-limit not of the above forms and $X \subset L_\alpha$:

$$L_s\{X\} = \bigcup \left\{L_t\{X\} \,\middle|\, t \mathrel{\widetilde{<}} s \text{ an } \alpha\text{-location}\right\}$$

Compactness Property Let s be an α-location and $X \subset L_\alpha$, then $x \in L_s\{X\}$ iff $x \in L_s\{Y\}$ for some finite $Y \subset X$.

Lemma 1 *Let s be a γ-location, $X \triangleleft L_s$, $\pi\colon X \cong L_{\bar{s}}$, and $t \mathrel{\widetilde{\leq}} s$, $t \in X \cup \{\gamma\}$, then (let $\alpha = \alpha(t)$):*

$$\forall Z \subset X \cap L_\alpha \, \pi[L_t\{Z\}] = L_{\pi(t)}\{\pi[Z]\},$$

Proof First note, that $X \cap L_\alpha \triangleleft L_t$. Hence $\pi \restriction X \cap L_\alpha \colon X \cap L_\alpha \cong L_{\bar{t}}$ where $\bar{t} = \widetilde{\leq}\text{-lub}\, \pi\left[\left\{r \mathrel{\widetilde{<}} t \,\middle|\, r \in X \cap L_\alpha \cup \{\alpha\}\right\}\right]$ (with $\pi(\gamma) = \alpha(\bar{s})$). Then, of course, $\pi[L_t\{Z\}] = L_{\bar{t}}\{\pi[Z]\}$ for $Z \subset X \cap L_\alpha$. It remains to show that $\bar{t} = \pi(t)$.

Since π preserves the $\widetilde{\leq}$-relation, $\bar{t} \mathrel{\widetilde{\leq}} \pi(t)$. On the other hand, let $r = (\beta, \varphi, \vec{b}) \mathrel{\widetilde{<}} \pi(t)$. Then $\beta \leq \pi(\alpha)$ and $\vec{b} \in L_\beta \subset L_{\pi(\alpha)} = \pi[X \cap \alpha]$. So there are $\delta \in X \cap L_\alpha \cup \{\alpha\}$ and $\vec{d} \in X \cap L_\alpha$ s.t. $r = (\pi(\delta), \varphi, \pi(\vec{c}))$. But then $\pi^{-1}(r) = (\delta, \varphi, \vec{c}) \mathrel{\widetilde{<}} t$ and $\pi^{-1}(r) \in X \cap L_\alpha \cup \{\alpha\}$. Therefore, by definition of $\bar{t}$ we have $r \mathrel{\widetilde{<}} \bar{t}$. $\dashv$

Next we fix our language for the investigation of morasses.

Definition 2 (Language $\mathcal{L}$ for L_s) Let s be an α-location. We take *function symbols* for the structure L_s discussed above: naming N, interpretation I, Skolem function S, location decomposition $\alpha(\cdot)$ and $\vec{x}(\cdot)$, and location composition $(\cdot, \varphi., \cdot)$. (The distinction between function symbols and functions won't be shown, same for relations etc.). We have *relation symbols* $\in$, $<_L$ (on sets, i.e., on elements of the structure), $=$ (on sets and locations) and $\widetilde{<}$, $\widetilde{\leq}$ (on locations). Finally, we have *variables* for sets.

Terms are defined as usual, note that there will be terms for sets and for locations: Variables are terms. If x, y are set terms or y is α (strictly speaking a constant symbol for the top level) and t a location term, then the following are also terms: $N(x)$, $I(t)$, $S(t)$, $\alpha(t)$, $\vec{x}(t)$, (y, φ_n, x) for $n < \omega$.

Interpretation of terms. Given a term t with variables v_i, $i < k$ for some $k < \omega$, interpreted as $a_i \in L_\alpha$. Then the interpretation t^s of t is defined inductively: If t is of the form v_i then $t^s = a_i$. If t is of the form (t_0, φ_n, t_1) and t_0^s is defined and an ordinal or α, $n < \omega$, and t_1^s is defined and a vector of length m of elements of $L_{t_0^s}$ where φ_n has $m+1$ free variables ("t_1 is of the right length"), then $t^s = (t_0^s, \varphi_n, t_1^s)$ provided that this is $\tilde{<}\ s$, else undefined. If $t^s = (\beta, \varphi_n, \vec{z})$ is defined with $\beta < \alpha$ then $\alpha(t)^s = \beta$ and $I(t)^s = I(t^s)$. If $t^s = (\beta, \varphi_n, \vec{z})$ is defined then $\vec{x}(t)^s = \vec{z}$ and $S(t)^s = S(t^s)$ (here $\beta \leq \alpha$). If t^s is defined and t a set term then $N(t)^s = N(t^s)$. All other terms are undefined, we write $t\uparrow$; also $t\downarrow$ iff $\neg\, t\uparrow$.

We say that *the term t is determined by location s* iff for each subterm of the form (α, φ_n, u) where u^s is defined, if (α, φ_n, u^s) is a location then it is $\tilde{<}\ s$.

Given set terms x_0, x_1 as well as location terms t_0, t_1 the following are *atomic formulas*: $x_0 \in x_1$, $x_0 <_L x_1$, $x_0 = x_1$, $t_0 = t_1$, $t_0 \tilde{<} t_1$ and $t_0 \tilde{\leq} t_1$. Atomic formulas are *formulas*. And if φ, χ are formulas and v is a variable, then $\varphi \wedge \chi$, $\neg\, \varphi$ and $\exists v\, \varphi(v)$ are formulas. A *quantifier-free formula (QFF)* is a formula with no occurrence of $\exists$. A Σ_1*-formula* is a formula of the form $\exists v\, \varphi(v)$ where φ is quantifier-free; instead of v a tuple $\vec{v}$ is allowed.

We say that *a formula φ (together with an assignment of its free variables) is determined by location s* iff each term in it is determined by location s.

Given an assignment of the variables, we define truth for a determined formula φ ($L_s \models \varphi$) as follows: Equality is true in L_s iff both sides are defined and equal or both sides are undefined. The other relations must have both sides defined to be true. $\varphi \wedge \chi$ is true iff φ and χ are true, $\neg\, \varphi$ is true iff φ is false and $\exists v\, \varphi(v)$ is true iff there is an $a \in L_\alpha$ s. t. $\varphi(a)$ holds.

The hull of X for the location s, $L_s\{X\}$, is the set of values of defined terms with parameters from X.

The Σ_1-*hull* of X for the location s is the closure of the normal hull $L_s\{X\}$ under $<_L$-least witnesses for Σ_1-formulas. We write $L_s^*\{X\}$.

Remark 3 *The following observations about our language are straightforward (t a term, φ a formula, given an assignment):*

Assume t is determined by s. Then so is every subterm of t. Further, if $s \mathrel{\widetilde{<}} s'$ with $\alpha(s) = \alpha(s')$ then t is determined by s'; also t^s is defined iff $t^{s'}$ is defined, in which case their values agree. If s is a limit location then t is already determined by a location $s' \mathrel{\widetilde{<}} s$ (note that if s is a minimal location with $\alpha(s)$ a successor ordinal, it will be formally necessary to replace terms interpreted as $\alpha(s')$, if any, by the constant symbol for the top level of $L_{s'}$); furthermore, s' can be taken from $L_s\{\vec{a}\}$ where $\vec{a}$ is assigned to the free variables of t. The latter implies that a structure-preserving map between structures L_s with s limit preserves the determinedness of terms.

If t is determined by s, t^s is defined, $s \mathrel{\widetilde{<}} s'$ with $\alpha(s) < \alpha(s')$, then t' is determined by s' with $t^s = (t')^{s'}$ where t' is the same as t with all references to the top level α replaced by $\alpha(s)$.

If t is determined by s, then $t\!\uparrow$ (and hence also $t\!\downarrow$) can be expressed by a QFF: If t is a set term we have $t\!\uparrow$ iff $t = S(0, y \in y, \emptyset)$; if t is a location term we have $t\!\uparrow$ iff $t = (0, \varphi_0, 1)$.

If $s \mathrel{\widetilde{<}} s'$ with $\alpha(s) = \alpha(s')$ and φ is quantifier-free and determined by s, then φ is determined by s' and $L_s \models \varphi$ iff $L_{s'} \models \varphi$.

The concept of "determined" is needed so that a term which is undefined cannot become defined for a bigger location on the same level, thereby changing truth values of formulas. For level changes we also get persistence provided terms are translated (as indicated above). From now on those translations won't be mentioned.

If $s \mathrel{\widetilde{<}} s'$ and φ is a Σ_1-formula with $L_s \models \varphi$, then $L_{s'} \models \varphi$.

If φ is a Σ_1-formula with $L_s \models \varphi$ and s is a limit location, then there is an $s' \mathrel{\widetilde{<}} s$ s. t. $L_{s'} \models \varphi$.

Let $\pi\colon L_s \to L_t$ be a structure-preserving map with s, t limit locations. π is Σ_1-preserving iff $\operatorname{range}\pi$ is Σ_1-closed (i. e., $\operatorname{range}\pi = L_t^*\{\operatorname{range}\pi\}$): Clearly if $\operatorname{range}\pi$ is Σ_1-closed then π is Σ_1-preserving; for the other direction just note that if you have a witness for a Σ_1-formula then it is Σ_1 to say there is a smaller one.

Lemma 4 *Let s be a location and $s_0 = (\alpha_0, \varphi_{n_0}, p_0) \mathrel{\widetilde{<}} s$. For every term in the language for L_{s_0} we have a QFF in the language for L_s (uniformly definable using α_0 and p_0 as parameters and the free variables of the term)* *which is true in L_s for an L_{s_0}-assignment of the variables iff the term is defined in L_{s_0} with the same assignment.*

Proof This is done by induction on the complexity of a term (everything is evaluated according to the assignment). We write $\mathrm{def}_{s_0}(t)$ for "t is a defined term in L_{s_0}". For a variable v_i, set terms x, y and a location term t we have:

— $L_s \models \mathrm{def}_{s_0}(v_i)$.

— $L_s \models \mathrm{def}_{s_0}(\alpha(t))$ iff $L_s \models \mathrm{def}_{s_0}(t) \wedge \alpha(t) < \alpha_0$.

— $L_s \models \mathrm{def}_{s_0}(\vec{x}(t))$ iff $L_s \models \mathrm{def}_{s_0}(t)$.

— $L_s \models \mathrm{def}_{s_0}((x, \varphi_n, y))$ iff $L_s \models \mathrm{def}_{s_0}(x) \wedge \mathrm{def}_{s_0}(y) \wedge (x, \varphi_n, y)\downarrow$.

— $L_s \models \mathrm{def}_{s_0}((\alpha_0, \varphi_n, y))$ iff
$L_s \models \mathrm{def}_{s_0}(y) \wedge (\alpha_0, \varphi_n, y)\downarrow \wedge (\alpha_0, \varphi_n, y) \mathrel{\tilde{<}} s_0$.

— $L_s \models \mathrm{def}_{s_0}(N(x))$ iff $L_s \models \mathrm{def}_{s_0}(x)$.

— $L_s \models \mathrm{def}_{s_0}(I(t))$ iff $L_s \models \mathrm{def}_{s_0}(t) \wedge \alpha(t) < \alpha_0$.

— $L_s \models \mathrm{def}_{s_0}(S(t))$ iff $L_s \models \mathrm{def}_{s_0}(t) \wedge S(t)\downarrow$. ⊣

A similar result holds not only as above for the property "t is determined", but also for the property "t is defined":

Lemma 5 *Let s be a location and $s_0 = (\alpha_0, \varphi_{n_0}, p_0) \mathrel{\tilde{<}} s$. For every term in the language for L_{s_0} we have a QFF in the language for L_s (uniformly definable using α_0 and p_0 as parameters and the free variables of the term) which is true in L_s for an L_{s_0}-assignment of the variables iff the term is determined by s_0 with the same assignment.*

Proof As in the previous lemma, this is done by induction on the complexity of a term where everything is evaluated according to the assignment. We write $\det_{s_0}(t)$ for "t is determined by s_0". For a variable v_i, set terms x, y and a location term t we have:

— $L_s \models \det_{s_0}(v_i)$.

— $L_s \models \det_{s_0}(\alpha(t))$ iff $L_s \models \det_{s_0}(t)$.

— $L_s \models \det_{s_0}(\vec{x}(t))$ iff $L_s \models \det_{s_0}(t)$.

— $L_s \models \det_{s_0}((x, \varphi_n, y))$ iff $L_s \models \det_{s_0}(x) \wedge \det_{s_0}(y)$.

— $L_s \models \det_{s_0}((\alpha_0, \varphi_n, y))$ iff
$L_s \models \det_{s_0}(y) \wedge \big((\alpha_0, \varphi_n, y){\downarrow} \rightarrow (\alpha_0, \varphi_n, y) \mathrel{\widetilde{<}} s_0\big)$.

— $L_s \models \det_{s_0}(N(x))$ iff $L_s \models \det_{s_0}(x)$.

— $L_s \models \det_{s_0}(I(t))$ iff $L_s \models \det_{s_0}(t)$.

— $L_s \models \det_{s_0}(S(t))$ iff $L_s \models \det_{s_0}(t)$. ⊣

Corollary 6 *Let s be a location and $s_0 = (\alpha_0, \varphi_{n_0}, p_0) \mathrel{\widetilde{<}} s$. For every QFF φ in the language for L_{s_0} there is a QFF φ' in the language for L_s (uniformly definable) which is true in L_s for an L_{s_0}-assignment of the variables iff φ is true in L_{s_0} with the same assignment.*

Proof Using $\det_{s_0}$ for every term in φ we can check that φ is determined by s_0. Then by induction on the complexity of the formula using def_{s_0} and $\det_{s_0}$ we express the truth of φ. ⊣

Definition 7 (Type) Let s be a location and $\vec{x}, \vec{p} \in L_{\alpha(s)}$. Define:

$$\begin{aligned}\mathrm{Type}(s, \vec{x}, \vec{p}) = \{(0, \tau_0, \tau_1) \mid \tau_0,\ \tau_1 \text{ terms, } L_s \models \tau_0(\vec{x}, \vec{p}) = \tau_1(\vec{x}, \vec{p})\} \cup \\ \cup \{(1, \tau_0, \tau_1) \mid \tau_0,\ \tau_1 \text{ terms, } L_s \models \tau_0(\vec{x}, \vec{p}) \in \tau_1(\vec{x}, \vec{p})\}\end{aligned}$$

Lemma 8 (Type Preservation) *Let $\pi\colon L_s \rightarrow L_t$ be a Σ_1-preserving map, $s_0 \mathrel{\widetilde{\leq}} s_1 \mathrel{\widetilde{\leq}} s$, $\vec{p_0} \in L_{s_0}$, $\vec{p_1} \in L_{s_1}$, s_0, s_1 limit locations, and $\alpha \leq \alpha(s_0)$. Then:*

$$\begin{aligned}&\forall \vec{x} \in \alpha\ \mathrm{Type}\,(s_0, \vec{x}, \vec{p_0}) = \mathrm{Type}\,(s_1, \vec{x}, \vec{p_1}) \ \textit{ iff} \\ &\forall \vec{x} \in \pi(\alpha)\ \mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) = \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))\end{aligned}$$

Proof $\forall$ is preserved downwards (note the implicit $\forall$ quantification over terms). So it remains to show, that the upward direction is preserved. Let

$$\neg\, \forall \vec{x} \in \pi(\alpha)\ \mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) = \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))$$

Equivalently:

$$\exists \vec{x} \in \pi(\alpha)\ \mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) \neq \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))$$

Hence there are terms τ_0, τ_1 which witness this inequality, e.g., $(0, \tau_0, \tau_1)$ is in the right Type but not in the left one. So using corollary 6 we can write:

$$\begin{aligned}L_t \models \exists \vec{x} \in \pi(\alpha)\ \big(L_{\pi(s_0)} \models (\tau_0(\vec{x}, \pi(\vec{p_0})) \neq \tau_1(\vec{x}, \pi(\vec{p_0}))) \wedge \\ \wedge\, L_{\pi(s_1)} \models (\tau_0(\vec{x}, \pi(\vec{p_1})) = \tau_1(\vec{x}, \pi(\vec{p_1})))\big)\end{aligned}$$

This is a Σ_1-statement and therefore preserved. ⊣

Corollary 9 *With the hypotheses of the lemma we get:*

$$L_{s_0}\{\alpha \cup \vec{p_0}\} \cong L_{s_1}\{\alpha \cup \vec{p_1}\} \text{ \textit{iff}}$$
$$L_{\pi(s_0)}\{\pi(\alpha) \cup \pi(\vec{p_0})\} \cong L_{\pi(s_1)}\{\pi(\alpha) \cup \pi(\vec{p_1})\}$$

Proof First assume $\pi_1\colon L_{s_0}\{\alpha \cup \vec{p_0}\} \cong L_{s_1}\{\alpha \cup \vec{p_1}\}$. π_1 is structure preserving and hence preserves determinedness of terms. Therefore, we have $\mathrm{Type}\,(s_0, \vec{x}, \vec{p_0}) = \mathrm{Type}\,(s_1, \vec{x}, \vec{p_1})$ for all $\vec{x} \in \alpha$. Now apply type preservation along π to get $\mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) = \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))$ for all $\vec{x} \in \pi(\alpha)$. This shows we have an isomorphism as required: $L_{\pi(s_0)}\{\pi(\alpha) \cup \pi(\vec{p_0})\} \cong L_{\pi(s_1)}\{\pi(\alpha) \cup \pi(\vec{p_1})\}$. The same argument works for the other direction. $\dashv$

Gap 1 Morasses in L

Definition 10 (Gap 1 Morass) An $(\omega_1, 1)$-*morass* (morass, from now on) is a structure of the form $\left\langle S^1, \multimap, (\pi_{\sigma\tau})_{\sigma \multimap \tau} \right\rangle$ with

$$\begin{array}{lll} S^0,\ S^1 & \subset & \omega_2, \\ \gamma_\sigma & \in & S^0 \text{ for } \sigma \in S^1, \\ S_\gamma & := & \left\{ \sigma \in S^1 \mid \gamma_\sigma = \gamma \right\} \text{ for } \gamma \in S^0, \\ S^0 & = & \left\{ \gamma_\sigma \mid \sigma \in S^1 \right\}, \text{ and} \\ \prec,\ \multimap & \subset & S^1 \times S^1. \end{array}$$

$\prec$, $\multimap$ are strict partial orderings on S^1.
$\prec$ is defined by $\prec = \bigcup_{\gamma \in S^0} (< \cap (S_\gamma \times S_\gamma))$.

(M0) i) $\forall \gamma \in S^0 \cap \omega_1\ S_\gamma$ closed

ii) S_{ω_1} club in ω_2

iii) $\omega_1 = \sup(S^0 \cap \omega_1) \in S^0$

iv) $\multimap$ is a tree-ordering on S^1

(M1) If $\sigma \multimap \tau$ then

i) $\pi_{\sigma\tau} \colon \sigma + 1 \to \tau + 1$, $\pi_{\sigma\tau} \restriction \gamma_\sigma = \mathrm{id} \restriction \gamma_\sigma$, $\gamma_\sigma < \pi_{\sigma\tau}(\gamma_\sigma) = \gamma_\tau$, $\pi_{\sigma\tau}(\sigma) = \tau$

ii) $\pi_{\sigma\tau}$ is order-preserving with $\pi_{\sigma\tau}^{-1}\left[S_{\gamma_\tau} \cap (\tau + 1)\right] = S_{\gamma_\sigma} \cap (\sigma + 1)$.

iii) For all $\nu \preceq \sigma$, ν is $\prec$-minimal, successor, limit iff $\pi_{\sigma\tau}(\nu)$ is $\prec$-minimal, successor, limit, respectively. In the successor case also the immediate predecessor is preserved.

(M2) Let $\sigma \multimap \tau$, $\bar\sigma \prec \sigma$, and $\bar\tau := \pi_{\sigma\tau}(\bar\sigma)$, then $\bar\sigma \multimap \bar\tau$ via $\pi_{\bar\sigma\bar\tau} = \pi_{\sigma\tau} \restriction (\bar\sigma + 1)$.

(M3) For $\tau \in S^1$ $\{\gamma_\sigma \mid \sigma \multimap \tau\}$ closed in γ_τ.

(M4) If τ is not $\prec$-maximal then $\{\gamma_\sigma \mid \sigma \multimap \tau\}$ cofinal in γ_τ.

(M5) If $\{\gamma_\sigma \mid \sigma \multimap \tau\}$ is unbounded in γ_τ, then $\tau = \bigcup\limits_{\sigma \multimap \tau} \pi_{\sigma\tau}[\sigma]$.

(M6) If $\sigma \multimap \tau$, σ a $\prec$-limit, and $\lambda := \sup \operatorname{range} \pi_{\sigma\tau} \restriction \sigma < \tau$, then $\sigma \multimap \lambda$ with $\pi_{\sigma\lambda} \restriction \sigma = \pi_{\sigma\tau} \restriction \sigma$.

(M7) If $\sigma \multimap \tau$, σ a $\prec$-limit, and $\tau = \sup \operatorname{range} \pi_{\sigma\tau} \restriction \sigma$, then for $\alpha \in S^0$, if $\forall \bar\sigma \prec \sigma\ \exists \bar\upsilon \in S_\alpha\ \bar\sigma \multimap \bar\upsilon \multimap \pi_{\sigma\tau}(\bar\sigma)$ then $\exists \upsilon \in S_\alpha\ \sigma \multimap \upsilon \multimap \tau$.

Definition 11 $\sigma < \omega_2$ is called $(\omega_1, 1)$-*morass point* (morass point, from now on) iff $\sigma = \bigcup \{\mu < \sigma \mid L_\mu \models ZF^-\}$, and $L_\sigma \models \exists! \gamma \in \text{Card } \gamma > \aleph_0$. In this case, let γ_σ be this unique ordinal. Let $S^1 = \{\sigma < \omega_2 \mid \sigma \text{ morass point}\}$ and $S^0 := \{\gamma_\sigma \mid \sigma \in S^1\}$. For $\sigma, \tau \in S^1$ define $\sigma \prec \tau$ iff $\sigma < \tau \wedge \gamma_\sigma = \gamma_\tau$.

For $\sigma \in S^1$ let $s(\sigma)$ be the $\widetilde{\leq}$-least location s s. t. there is a $p \in {}^{<\omega}L_{\alpha(s)}$ with $L_s \{\gamma_\sigma \cup p\} \cap \sigma$ cofinal in σ (we say: $L_s \{\gamma_\sigma \cup p\}$ collapses σ); in this case let p_σ be the $<^*$-least such. Note that $s(\sigma)$ is a limit location by the Finiteness Property.

Define the partial ordering $\multimap$ on S^1 by letting $\sigma \multimap \tau$ iff there exists $\pi\colon L_{s(\sigma)} \to L_{s(\tau)}$ with:

i) π is Σ_1-preserving.

ii) $\pi \restriction \gamma_\sigma = \text{id} \restriction \gamma_\sigma$, $\gamma_\sigma < \pi(\gamma_\sigma) = \gamma_\tau$, $\tau = \pi(\sigma)$, $p_\tau \in \text{range}\, \pi$
(define $\pi(\sigma) = \tau$ if $\sigma \notin \text{dom}\, \pi$)

iii) If τ is a $\prec$-successor with immediate predecessor τ', then $\tau' \in \text{range}\, \pi$.

Lemma 12

i) $\sigma \subset L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\}$

ii) $L_{s(\sigma)} \{\sigma \cup p_\sigma\} = L_{s(\sigma)}$

iii) $L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\} = L_{s(\sigma)}$

iv) The map $\pi\colon L_{s(\sigma)} \to L_{s(\tau)}$, *if exists, is uniquely determined.*

v) $\pi(p_\sigma) = p_\tau$

vi) $\pi[S_{\gamma_\sigma} \cap (\sigma + 1)] \subseteq S_{\gamma_\tau} \cap (\tau + 1)$. *If* τ' *is the immediate* $\prec$*-predecessor of* τ, *then* $\pi^{-1}(\tau')$ *is the immediate* $\prec$*-predecessor of* σ.

Proof For i) assume $\xi \in L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\} \cap \sigma$. Let $\eta \in L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ s. t. $\exists f \in L_\eta\, f\colon \gamma_\sigma \leftrightarrow \xi$. In particular, $S\,(\eta, v_0\colon \aleph_1 \leftrightarrow v_1, \langle \xi \rangle)$ is such a map. Therefore, $\xi = \text{range}\, f \subset L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\}$. Using that the hull is cofinal in σ we have that σ actually is a subset.

For ii) consider $L_{s(\sigma)} \{\sigma \cup p_\sigma\} \cong L_{\bar{s}} \{\sigma \cup \bar{p}\} = L_{\bar{s}}$. Then $\bar{s}, \bar{p}$ satisfy the definition of $s(\sigma), p_\sigma$; by minimality we have $s(\sigma) = \bar{s}$ and $p_\sigma = \bar{p}$.

iii) follows from i) and ii). Now iv) is clear.

For v) note, that $p_\tau \in \text{range}\, \pi$. By definition $\pi(p_\sigma) \in L_{s(\tau)} \{\gamma_\tau \cup p_\tau\}$. Using Σ_1-preservation, we get $p_\sigma \in L_{s(\sigma)} \{\gamma_\sigma \cup \pi^{-1}(p_\tau)\}$ and hence $L_{s(\sigma)} =$

$L_{s(\sigma)}\left\{\gamma_\sigma \cup \pi^{-1}(p_\tau)\right\}$. Therefore, $p_\sigma \leq^* \pi^{-1}(p_\tau)$. Assume for contradiction that this is strict. Then we get $\pi(p_\sigma) <^* p_\tau$. But $p_\tau \in L_{s(\tau)}\left\{\gamma_\sigma \cup \pi(p_\sigma)\right\} \subset L_{s(\tau)}\left\{\gamma_\tau \cup \pi(p_\sigma)\right\} = L_{s(\tau)}$ contradicting the minimality of p_τ.

For vi) suppose that $\sigma' \prec \sigma$; we must show that $\pi(\sigma')$ is a morass point. Let $\eta < \sigma$ be large enough s. t. $L_\eta \models ZF^- \wedge \sigma'$ morass point. Now $\pi \upharpoonright L_\eta$ is elementary and therefore $L_{\pi(\eta)} \models \pi(\sigma')$ morass point; therefore $\pi(\sigma')$ is a morass point. The second part of vi) now follows from the first part and requirement iii) on the map π. $\dashv$

Definition 13 (morass map) For $\sigma \mathrel{-\!\!3} \tau$, let $\pi_{\sigma\tau}$ be the unique map from the previous lemma. The actual morass map to satisfy the morass axioms will be $\pi_{\sigma\tau} \upharpoonright (\sigma+1)$, but we will write $\pi_{\sigma\tau}$ for both maps and work with the underlying map only.

Theorem 14 $\left\langle S^1, \mathrel{-\!\!3}, (\pi_{\sigma\tau})_{\sigma \mathrel{-\!\!3} \tau}\right\rangle$ *as defined above is an* $(\omega, 1)$*-morass.*

Proof For **(M0)** the first three assertions are clear. To see that $\mathrel{-\!\!3}$ forms a tree-ordering, presume $\pi_{\sigma\tau}$ and $\pi_{\upsilon\tau}$ are morass maps with $\sigma < \upsilon$; then $\pi_{\sigma\upsilon} = \pi_{\upsilon\tau}^{-1} \circ \pi_{\sigma\tau}$ is as required.

For **(M1)** the first assertion is as defined. For ii) note first that by the first part of Lemma 12 vi), morass points $\bar{\nu} \leq \sigma$ are mapped to morass points $\leq \tau$. Clearly, the map is order-preserving. The next properties for morass points below the top are immediate by elementarity. For the morass point at the top use the second part of Lemma 12 vi).

To see **(M2)** first note that by (M1) $\bar{\tau}$ is a morass point. Using that L_σ is a limit of ZF^--models find $\eta < \sigma$ s. t. $L_{s(\bar{\sigma})}$ and $p_{\bar{\sigma}}$ are definable in L_η from the parameter $\bar{\sigma}$. Hence $\pi_{\sigma\tau} \upharpoonright L_\eta$ is elementary and, therefore, maps $L_{s(\bar{\sigma})}$ into $L_{s(\bar{\tau})}$ and $p_{\bar{\sigma}}$ onto $p_{\bar{\tau}}$. Then $\pi_{\bar{\sigma}\bar{\tau}}$ is as required.

For **(M3)** let $\tau \in S^1$ and $\bar{\alpha} < \gamma_\tau$ a limit point of $\{\gamma_\sigma \mid \sigma \mathrel{-\!\!3} \tau\}$. By condensation let $\pi \colon L_{s(\tau)}\left\{\bar{\alpha} \cup p_\tau\right\} \cong L_{\bar{s}}$ and $\bar{\tau} = \pi(\tau)$, $\bar{p} = \pi(p_\tau)$. Note that $L_{s(\tau)}\left\{\bar{\alpha} \cup p_\tau\right\} \cap \gamma_\tau = \bar{\alpha}$, since $\bar{\alpha}$ is the limit of $L_{s(\tau)}\left\{\gamma_\sigma \cup p_\tau\right\} \cap \gamma_\tau = \gamma_\sigma < \bar{\alpha}$.

We show $\bar{s} = s(\bar{\tau})$: Clearly $s(\bar{\tau}) \overset{\sim}{\leq} \bar{s}$, since $L_{\bar{s}} = L_{\bar{s}}\left\{\bar{\alpha} \cup \bar{p}\right\}$ cofinal in $\bar{\tau}$. Now assume for contradiction that $s(\bar{\tau}) \overset{\sim}{<} \bar{s}$. Let $\pi_\sigma = \pi \circ \pi_{\sigma\tau}$ for $\sigma \in \{\sigma \mathrel{-\!\!3} \tau \mid \gamma_\sigma < \bar{\alpha}\}$. Choose σ large enough s. t. exist $\tilde{s}, \tilde{p} \in L_{s(\sigma)}$ with $s(\bar{\tau}) = \pi_\sigma(\tilde{s})$ and $p_{\bar{\tau}} = \pi_\sigma(\tilde{p})$. By $s(\bar{\tau}) \overset{\sim}{<} \bar{s}$ we have $\tilde{s} \overset{\sim}{<} s(\sigma)$ and hence $L_{\tilde{s}}\left\{\gamma_\sigma \cup \tilde{p}\right\}$ bounded in σ, say by β. But this bound is preserved by $\pi_{\sigma\tau}$ and by π (hence by π_σ);

therefore, we get that $L_{s(\bar\tau)}\{\bar\alpha \cup p_{\bar\tau}\} \cap \bar\tau$ is bounded by $\pi_\sigma(\beta) < \bar\tau$ which contradicts the definition of $s(\bar\tau)$ and $p_{\bar\tau}$.

To see that $\pi^{-1}\colon L_{s(\bar\tau)} \to L_{s(\tau)}$ is a morass map and hence $\bar\tau \multimap \tau$ with $\gamma_{\bar\tau} = \bar\alpha$, we need to show, that π^{-1} preserves Σ_1; the other properties follow by definition, for p_τ and the predecessor of τ (if any) note that $\operatorname{dom}\pi$ contains the ranges of morass maps as subsets.

As a collapsing map, π^{-1} is structure-preserving. Σ_1 is preserved upwards. Now assume, we have a Σ_1-formula in $L_{s(\tau)}$. It is preserved downwards by morass maps $\pi_{\sigma\tau}$ for $\sigma \in \{\sigma \multimap \tau \mid \gamma_\sigma < \bar\alpha\}$ and hence has a witness in $\operatorname{range}\pi_{\sigma\tau} \subset \operatorname{dom}\pi$.

For the proof of **(M4)** let $\upsilon \in S_{\gamma_\tau}$ with $\tau < \upsilon$. Let $\alpha < \gamma_\tau$ be arbitrary and η between τ and υ s.t. $L_{s(\tau)} \in L_\eta$ and $L_\eta \models ZF^-$. Let $X \prec L_\eta$ s.t. $L_{s(\tau)}\{\alpha \cup p_\tau\} \cup \{\tau\} \subset X$ and $\bar\alpha := X \cap \gamma_\tau \in \gamma_\tau$. Let $\pi\colon X \cong L_{\bar\eta}$, $\sigma = \pi(\tau)$, and $\bar p = \pi(p_\tau)$. So σ is a morass point and $\pi^{-1} \restriction L_{s(\sigma)}\colon L_{s(\sigma)} \to L_{s(\tau)}$ is elementary and, therefore, a morass map. Hence $\sigma \multimap \tau$ and $\alpha \le \gamma_\sigma = \bar\alpha$.

For **(M5)** consider $\xi \in \tau \in S^1$ and $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\}$. By cofinality there exists a $\sigma \multimap \tau$ with $\xi \in L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} = \operatorname{range}\pi_{\sigma\tau}$.

For **(M6)** let $\tilde s = \tilde\le$-lub$\{\pi_{\sigma\tau}(t) \mid t \mathrel{\tilde<} s(\sigma)\}$. We show $L_{\tilde s}\{\gamma_\tau \cup p_\tau\} \cap \tau = \lambda$: First assume $\lambda_0 \in \lambda$; then there is λ_1 with $\lambda_0 < \lambda_1 < \lambda$ and $\lambda_1 = \pi_{\sigma\tau}(\bar\lambda_1)$. Then $L_\sigma \models \operatorname{card}\bar\lambda_1 \le \gamma_\sigma$, hence there exists $\bar f \in L_\sigma$ s.t. $\bar f\colon \gamma_\sigma \to \bar\lambda_1$ is onto, in particular $\bar f \in L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$. As $s(\sigma)$ is a limit location, we have $\bar f \in L_t\{\gamma_\sigma \cup p_\sigma\}$ for some $t \mathrel{\tilde<} s(\sigma)$. Let $f = \pi_{\sigma\tau}(\bar f) \in L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\}$, then $f\colon \gamma_\tau \to \lambda_1$ is onto, so $\lambda_0 \in \operatorname{range} f$, hence $\lambda_0 \in L_{\tilde s}\{\gamma_\tau \cup p_\tau\}$. On the other hand assume $\lambda_0 \in L_{\tilde s}\{\gamma_\tau \cup p_\tau\} \cap \tau$, then there is a $t \mathrel{\tilde<} s(\sigma)$ s.t. $\lambda_0 \in L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\}$. But $L_t\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say), since $t \mathrel{\tilde<} s(\sigma)$, hence also $L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ, namely by $\pi_{\sigma\tau}(\beta) < \lambda$. So $\lambda_0 \in \lambda$ as required.

Let $\pi\colon L_{\tilde s}\{\gamma_\tau \cup p_\tau\} \cong L_{s_0}$ and $p_0 = \pi(p_\tau)$ (then $\lambda = \pi(\tau)$). Note that $\lambda \in S_{\gamma_\tau}$. We show $L_{s_0}\{\gamma_\tau \cup p_0\} = L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$:

$s_0 = s(\lambda)$: First note that s_0 singularizes λ, so $s(\lambda) \mathrel{\tilde\le} s_0$. Assume for contradiction that s_0 is strictly greater. As $p_\lambda \in L_{s_0}\{\gamma_\tau \cup p_0\}$, we have $p_\lambda \in L_{s_1}\{\gamma_\tau \cup p_0\}$ where $s(\lambda) \mathrel{\tilde<} s_1 \mathrel{\tilde<} s_0$ (and where $\alpha(s(\lambda))$ belongs to $L_{s_1}\{\gamma_\tau \cup p_0\}$ in case $\alpha(s(\lambda)) < \alpha(s_0)$; of course we are using the fact that s_0 is a limit location). Since $L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\} \subset L_{s_1}\{\gamma_\tau \cup p_0\}$, s_1 singularizes

λ. By definition of s_0, $\pi^{-1}(s_1) \mathrel{\widetilde{<}} \tilde{s}$. Further, by definition of $\tilde{s}$, there is a $t \mathrel{\widetilde{<}} s(\sigma)$ s. t. $\pi^{-1}(s_1) \mathrel{\widetilde{\leq}} \pi_{\sigma\tau}(t)$. By minimality of $s(\sigma)$, $L_t\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say). Hence $L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (by $\pi_{\sigma\tau}(\beta)$). Since $\pi^{-1}(s_1) \mathrel{\widetilde{\leq}} \pi_{\sigma\tau}(t)$, $L_{\pi^{-1}(s_1)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (still by $\pi_{\sigma\tau}(\beta)$). Apply π: $L_{s_1}\{\gamma_\tau \cup p_0\} \cap \lambda$ is bounded below λ (by $\pi \circ \pi_{\sigma\tau}(\beta)$), contradiction.

$p_0 = p_\lambda$: $L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_0\}$ is cofinal in λ (as above using $s_0 = s(\lambda)$). Therefore, $p_\lambda \leq^* p_0$. Assume for contradiction that p_0 is strictly greater, then using $p_0 \in L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$ and applying π^{-1} we get $\pi^{-1}(p_\lambda) <^* p_\tau \in L_{\tilde{s}}\left\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\right\} \subset L_{s(\tau)}\left\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\right\}$. Therefore, $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\} = L_{s(\tau)}\left\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\right\}$ contradicting the minimality of p_τ.

Let $\pi_0 = \pi \circ \pi_{\sigma\tau} \colon L_{s(\sigma)} \to L_{s(\lambda)}$. π_0 is well-defined as $\operatorname{range} \pi_{\sigma\tau} = L_{\tilde{s}}\{\gamma_\sigma \cup p_\tau\} \subset \operatorname{dom} \pi$. Further, $\pi_0(\sigma) = \lambda$ and $\pi_0(p_\sigma) = p_\lambda$. Since λ is a $\prec$-limit, property iii) of the morass map definition is vacuous. Finally, π_0 is Σ_1-preserving: First note that π_0 is structure-preserving. Σ_1 formulas are preserved by π_0 upwards, by π upwards (from $L_{s(\lambda)}$ to $L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$), and by $\pi_{\sigma\tau}$ downwards, hence by π_0 both ways. Now $\pi_0 = \pi_{\sigma\lambda}$ is a morass map, hence $\sigma \multimap \lambda$ as required.

For **(M7)** we first show that $L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau = \alpha$, clearly α is a subset of the left side. For the other direction note that since we assume $\tau = \sup \operatorname{range} \pi_{\sigma\tau} \restriction \sigma$, the argument for (M6) shows that $s(\tau) = \widetilde{\leq}$-lub $\left\{\pi_{\sigma\tau}(t) \,\middle|\, t \mathrel{\widetilde{<}} s(\sigma)\right\}$. Let $\xi \in L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$, then there is $s_0 \mathrel{\widetilde{<}} s(\sigma)$ s. t. $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Working downstairs we have that $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ does not collapse σ (by minimality of $s(\sigma) \mathrel{\widetilde{>}} s_0$). Let $\pi_0 \colon L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\} \cong L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ where $\bar{p} = \pi_0^{-1}(p_\sigma)$. Then $\sigma' := \pi_0^{-1}(\sigma) < \sigma$. $L_{\bar{s}}$ cannot collapse σ', else there would be a map from γ_σ onto σ' and hence a map from γ_σ onto σ in $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$. Therefore, $L_{\bar{s}} \models \operatorname{Card} \sigma'$ and $L_\sigma \models \neg \operatorname{Card} \sigma'$, hence $L_{\bar{s}} \in L_\sigma$. Now, σ is a $\prec$-limit, so there is $\bar{\sigma} \prec \sigma$ s. t. $L_{\bar{s}}, \bar{p} \in L_{s(\bar{\sigma})} = L_{s(\bar{\sigma})}\{\gamma_\sigma \cup p_{\bar{\sigma}}\}$.

Using lemma 8 (type preservation) we shift the isomorphism π_0 to $L_{s(\tau)}$:

$$\text{“}\pi_{\sigma\tau}(\pi_0)\text{”} \colon L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\} \cong L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$$

We started with $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Now we apply the isomorphism and infer $\xi \in L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau$ (since $\xi < \gamma_\tau$ it is not moved). Further, $L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau \subset L_{s(\pi_{\sigma\tau}(\bar{\sigma}))}\left\{\alpha \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\right\} \cap \gamma_\tau = \alpha$, where the former holds since $\pi_{\sigma\tau}(\bar{p}) \in L_{\pi_{\sigma\tau}(\bar{\sigma})}\left\{\gamma_\sigma \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\right\}$ and $\pi_{\sigma\tau}(\bar{s}) \mathrel{\tilde{<}} s(\pi_{\sigma\tau}(\bar{\sigma}))$ and the latter holds by $\bar{\sigma} \mathrel{-\!\!3} \bar{\upsilon} \mathrel{-\!\!3} \pi_{\sigma\tau}(\bar{\sigma})$ for some $\bar{\upsilon} \in S_\alpha$. Hence $\xi \in \alpha$ as desired.

Now we define $\pi \colon L_{s(\tau)}\{\alpha \cup p_\tau\} \cong L_{s'}\{\alpha \cup p'\} = L_{s'}$ where $p' := \pi(p_\tau)$, $\upsilon := \pi(\tau)$. By the previous argument we have $\pi^{-1}(\alpha) = \gamma_\tau$. Using the system of morass maps we have $\upsilon \in S_\alpha$.

We have to show $s' = s(\upsilon)$: $L_{s'} = L_{s'}\{\alpha \cup p'\}$ collapses υ, hence $s(\upsilon) \mathrel{\tilde{\leq}} s'$. Assume for a contradiction that $s(\upsilon) \mathrel{\tilde{<}} s'$. Since $p_\upsilon \in L_{s'}$ we have that there is an s_0 s. t. $s(\upsilon) \mathrel{\tilde{\leq}} s_0 \mathrel{\tilde{<}} s'$ and $p_\upsilon \in L_{s_0}\{\alpha \cup p'\}$. Since $\pi_{\sigma\tau}$ and π map locations cofinally this is also true for $\pi_0 := \pi \circ \pi_{\sigma\tau}$ (locations $\tilde{<} s(\sigma)$ are mapped to locations $\tilde{<} s'$). Hence without loss of generality, $s_0 = \pi_0(\bar{s}_0)$ where $\bar{s}_0 \mathrel{\tilde{<}} s(\sigma)$. Therefore, $L_{s(\sigma)} \models$ "$L_{\bar{s}_0}\{\gamma_\sigma \cup p_\sigma\}$ is bounded below σ". This is preserved by $\pi_{\sigma\tau}$: $L_{s(\tau)} \models$ "$L_{\pi_{\sigma\tau}(\bar{s}_0)}\{\gamma_\tau \cup p_\tau\}$ is bounded below τ". Finally, this is preserved by π downwards: $L_{s'} \models$ "$L_{s_0}\{\alpha \cup p'\}$ is bounded below υ", contradicting the definition of $s(\upsilon) \mathrel{\tilde{\leq}} s_0$.

Finally, we have to show that π^{-1} is Σ_1-preserving, then $\pi^{-1} = \pi_{\upsilon\tau}$ and $\pi_{\sigma\upsilon} = \pi_{\upsilon\tau}^{-1} \circ \pi_{\sigma\tau}$. First note that π is structure-preserving.

Σ_1 is preserved upwards by π^{-1} (i.e., from $L_{s(\upsilon)}$ to $L_{s(\tau)}\{\alpha \cup p_\upsilon\}$). For the other direction, assume $L_{s(\tau)} \models \exists x\, \varphi(x, \vec{r})$, where φ is quantifier-free and $\vec{r} \in \operatorname{dom} \pi = L_{s(\tau)}\{\gamma_\upsilon \cup p_\tau\}$; we have to show $L_{s(\upsilon)} \models \exists x\, \varphi(x, \pi(\vec{r}))$. As before, fix $s_0 \mathrel{\tilde{<}} s(\sigma)$ s. t. $\vec{r} \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\upsilon \cup p_\tau\}$ and $w \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$ where w is the least witness for $\exists x\, \varphi(x, \vec{r})$. Our aim is to show that γ_τ can be replaced by γ_υ in the latter hull.

Let $\pi_1 \colon L_{s_0}\{\gamma_\sigma \cup p_\sigma\} \cong L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$ where $\bar{p} = \pi_1(p_\sigma)$. As above using type preservation, we shift π_1 to the γ_τ-level, let's call the resulting map $\pi_2 \colon L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\} \cong L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$. Then we have $\pi_2(\vec{r}) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\upsilon \cup \pi_{\sigma\tau}(\bar{p})\}$ and $\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$: $L_{\pi_{\sigma\tau}(\bar{s})} \models \varphi(\pi_2(w), \pi_2(\vec{r}))$

Further, also as above, there is a $\bar{\sigma} \prec \sigma$ s. t. $L_{\bar{s}} \in L_{\bar{\sigma}}$ with $\bar{\sigma} \mathrel{-\!\!3} \bar{\upsilon} \mathrel{-\!\!3} \bar{\tau} := \pi_{\sigma\tau}(\bar{\sigma})$ and $\pi_2(\vec{r}), \pi_{\sigma\tau}(\bar{s}), \pi_{\sigma\tau}(\bar{p}) \in \operatorname{range} \pi_{\bar{\upsilon}\bar{\tau}}$. Therefore, $\pi_2(w) \in \operatorname{range} \pi_{\bar{\upsilon}\bar{\tau}}$ and hence by $\pi_{\bar{\upsilon}\bar{\tau}}$ being a morass map, we can replace γ_τ by γ_υ in "$\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$". Applying π_2^{-1} we get $w \in \operatorname{range} \pi_{\upsilon\tau}$. This proves Σ_1-preservation. $\dashv$

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