Δ_1 Definability of the Nonstationary Ideal

For uncountable regular κ, NS_κ denotes the ideal of nonstationary subsets of κ

Proposition

NS_κ is Σ_1 definable with parameter κ.

Proof. $X \in \mathrm{NS}_\kappa$ iff X is a subset of κ and there exists C such that C is a closed unbounded subset of κ disjoint from X.
This is Σ_1 with parameter κ. □

We say that NS_κ is Δ_1 *definable* if it is both Σ_1 and Π_1 definable using subsets of κ as parameters.

Δ_1 Definability of the Nonstationary Ideal

For NS_κ to be Δ_1 definable one needs to "witness stationarity". Typically this is not possible:

Theorem

Assume $V = L$. Then NS_κ is not Δ_1 definable.

Proof Sketch. Suppose that $\varphi(X)$ is a Σ_1 formula with a variable X denoting a subset of κ.

If $\varphi(X)$ is true then by condensation, $\varphi(X \cap \alpha)$ is true for club-many $\alpha < \kappa$; in fact, for club-many $\alpha < \kappa$, $\varphi(X \cap \alpha)$ is true "while α still looks regular", i.e. in some $L_\beta \vDash \alpha$ regular.

Conversely, if $\varphi(X)$ is false then for any club C there is α in C such that $\varphi(X \cap \alpha)$ is false in the largest $L_\beta \vDash \alpha$ regular.

So the club filter is "complete" for Σ_1 subsets of $\mathcal{P}(\kappa)$ and therefore not Δ_1. $\square$

Δ_1 Definability of the Nonstationary Ideal

Large cardinals also prevent NS_κ from being Δ_1 definable.

Theorem

Suppose that κ is weakly compact. Then NS_κ is not Δ_1 definable.

Proof. Again let $\varphi(X)$ be a Σ_1 formula with a variable X denoting a subset of κ.

As before, if $\varphi(X)$ is true then by condensation, $\varphi(X \cap \alpha)$ is true for club-many $\alpha < \kappa$.

Conversely, suppose that $\varphi(X)$ is false.
Then $\varphi(X)$ is false in $H(\kappa^+)$ and the latter is a Π^1_1 statement about V_κ. By weak compactness ($= \Pi^1_1$ reflection), $\varphi(X \cap \alpha)$ is false for stationary-many $\alpha < \kappa$.

So again the club filter is "complete" for Σ_1 subsets of $\mathcal{P}(\kappa)$ and therefore not Δ_1. $\square$

Δ_1 Definability of the Nonstationary Ideal

However it is indeed possible for NS_{ω_1} to be Δ_1 definable.

Theorem

(Mekler-Shelah, proof repaired by Hyttinen-Rautila) Assume GCH. Then there is a proper, cardinal-presering forcing extension satisfying GCH in which NS_{ω_1} is Δ_1 definable.

Idea of Proof. For $X \subseteq \omega_1$ let $T(X)$ be the tree of countable, closed subsets of X ordered by end-extension.
Then X contains a club iff $T(X)$ has a branch of length ω_1.
The idea is to force a tree T (called a canary tree) of size and height ω_1 with no ω_1-branch such that whenever X is stationary, costationary there are embeddings of $T(X)$ and $T(\sim X)$ into T.
Then conversely, if there are embeddings of both $T(X)$ and $T(\sim X)$ into T it follows that X is both stationary and costationary. So we have:

Δ_1 Definability of the Nonstationary Ideal

X is stationary iff
X contains a club or there are embeddings of both $T(X)$ and $T(\sim X)$ into T

and therefore NS_{ω_1} is Δ_1 definable. $\square$

With some extra work, Hyttinen-Rautila obtained the natural generalisation to NS_{κ^+} for any regular κ:

Let $\mathrm{Cof}(\kappa)$ denote the class of ordinals of cofinality κ and $NS_{\kappa^+} \restriction \mathrm{Cof}(\kappa)$ the ideal of stationary subsets of $\kappa^+ \cap \mathrm{Cof}(\kappa)$,

Theorem

(Hyttinen-Rautila) Assume GCH and κ regular. Then there is a κ-proper, cardinal-preserving forcing extension satisfying GCH in which $NS_{\kappa^+} \restriction \mathrm{Cof}(\kappa)$ is Δ_1 definable.

Δ_1 Definability of the Nonstationary Ideal

With a different strategy the Hyttinen-Rautila result can be improved.
For stationary $A \subseteq \kappa^+$ let $\mathrm{NS}_{\kappa^+} \restriction A$ denote the ideal of nonstationary subsets of A.

Theorem

(SDF-Hyttinen-Kulikov) Assume GCH and κ regular. Then for any costationary $A \subseteq \kappa^+$ there is a cardinal-preserving forcing extension satisfying GCH which preserves stationary subsets of A in which $NS_{\kappa^+} \restriction A$ is Δ_1 definable.

The difference now is that only stationary subsets of A, and not of $\sim A$, are preserved.
Thus the idea of the proof is to witness the stationarity of subsets of A by selectively killing the stationarity of certain "canonically chosen" subsets of $\sim A$ (obtained via a generic $\diamondsuit$ sequence).

Δ_1 Definability of the Nonstationary Ideal: Main Result

Obviously the strategy of making $\mathrm{NS}_{\kappa^+} \restriction A$ Δ_1 definable by killing the stationarity of subsets of $\sim A$ is of no use if one wants to obtain the Δ_1 definability of the full unrestricted NS_{κ^+}.

So a new idea is needed to show (our main result):

Theorem

(SDF-Wu-Zdomskyy) Assume $V = L$ and let λ be any infinite cardinal. Then there is a cardinal-preserving forcing extension satisfying GCH which preserves stationary subsets of λ^+ in which NS_{λ^+} is Δ_1 definable.

Thus we can handle the full NS at all successor cardinals.

I'll give now an outline of the proof.

Δ_1 Definability of the Nonstationary Ideal: Main Result

Let κ denote λ^+. We want to perform an iteration of length κ^+ which preserves the stationarity of subsets of κ, preserves cardinals and produces "witnesses" to the stationarity of subsets of κ.
Note that by Löwenheim-Skolem, if a subset of $\mathcal{P}(\kappa)$ is Σ_1 with a subset of κ as parameter then it is Σ_1 over $H(\kappa^+)$ and therefore our witnesses should be elements of $H(\kappa^+)$.

In fact the only parameter we will need is κ and our witnesses will be subsets of κ.

Now suppose that S is a stationary subset of κ and we want to "witness" this fact. The approach of SDF-Hyttinen-Kulikov was to fix a sequence $(S_i \mid i < \kappa^+)$ of "canonical" stationary subsets of κ and arrange that for some $\alpha < \kappa^+$, the stationarity of the S_i for i in $[\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa)$ is selectively killed so as to code S. But we can't do this as we want to preserve the stationarity of subsets of κ.

Δ_1 Definability of the Nonstationary Ideal: Main Result

So instead we choose "canonical" *stationary subsets* $(S_i \mid i < \kappa^+)$ *of* κ^+ (concentrating on $\mathrm{Cof}(\kappa)$) and arrange that for some $\alpha < \kappa^+$, the stationarity of the S_i for i in $[\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa)$ is selectively killed so as to code S.

But now our witnesses are subsets of κ^+ instead of κ so we only get a definition of the collection of stationary subsets of κ which is Σ_1 *over* $H(\kappa^{++})$ *with* κ^+ *as parameter*.

How do we convert this into a Σ_1 definition over $H(\kappa^+)$ with κ as parameter?

Here we use *localisation* (*David's trick*).

Δ_1 Definability of the Nonstationary Ideal: Main Result

Instead of just the "global property"

$S \subseteq \kappa$ is stationary iff S is coded into the stationarity of the $S_i \subseteq \kappa^+$ for i in $[\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa)$ for some $\alpha < \kappa^+$

we also ensure its "local version"

$S \subseteq \kappa$ is stationary iff for some $X \subseteq \kappa$, every "suitable" model M of size $< \kappa$ containing $X \cap \kappa^M$ (where κ^M denotes $(\lambda^+)^M$) satisfies that $S \cap \kappa^M$ is coded into the stationarity of the S_i^M for i in $[\kappa^M \cdot \alpha, \kappa^M \cdot \alpha + \kappa^M)$ for some $\alpha < (\kappa^M)^+$,
where $(S_i^M \mid i < ((\kappa^M)^+)^M)$ is M's version of $(S_i \mid i < \kappa^+)$.

The local version implies the global one by Löwenheim-Skolem and moreover yields a definition of stationarity for subsets of κ which is Σ_1 over $H(\kappa^+)$, as needed.

Δ_1 Definability of the Nonstationary Ideal: Main Result

In the local version

$S \subseteq \kappa$ is stationary iff for some $X \subseteq \kappa$, every "suitable" model M of size $< \kappa$ containing $X \cap \kappa^M$ satisfies that $S \cap \kappa^M$ is coded into the stationarity of the S_i^M for i in $[\kappa^M \cdot \alpha, \kappa^M \cdot \alpha + \kappa^M)$ for some $\alpha < (\kappa^M)^+$.

we say that X is a "local witness" (or "locally witnesses") that $S \subseteq \kappa$ is stationary.

We produce such a local witness X in three steps:

Δ_1 Definability of the Nonstationary Ideal: Main Result

1. Localise below κ^+, i.e. produce $Y \subseteq \kappa^+$ such that every "suitable" model M of size κ containing $Y \cap (\kappa^+)^M$ satisfies that S is coded into the stationarity of the $S_i^M = S_i \cap (\kappa^+)^M$ for i in $[\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa)$ for some $\alpha < \kappa^+$.

This is easy and does not require forcing.

2. Almost disjoint code Y into a subset X_0 of κ.

Then X_0 also localises below κ^+ as in 1.

3. Add the desired $X \subseteq \kappa$ satisfying $\text{Even}(X) = X_0$ by forcing with initial segments of length less than κ.

The fact that X_0 localises below κ^+ is sufficient to argue that this forcing is κ-distributive.

Δ_1 Definability of the Nonstationary Ideal: Main Result

Now I can describe the iteration $P = (P_\xi, \dot{Q}_\xi \mid \xi < \kappa^+)$.

In κ^+ steps, choose via bookkeeping names for stationary subsets S of κ, code such S by killing the stationarity of selected canonical stationary subsets S_i of κ^+ and localise these stationary-kills, thereby producing local witnesses to the stationarity of each stationary subset S of κ.

The iteration uses supports of size κ for killing the stationarity of selectd S_i's and supports of size less than κ for the localisation forcings.

There are three things to check about the iteration:

Δ_1 Definability of the Nonstationary Ideal: Main Result

1. The iteration is κ-distributive.

We show that P_ξ is κ-distributive by induction on $\xi \leq \kappa^+$.
Of course the induction hypothesis is stronger than this; we need to know that we can build conditions which serve as strong master conditions for each model in a sequence of models of length $\lambda + 1$ built by taking successive Skolem hulls. So the argument is Jensen-style, tracing back to his coding work, and not Shelah-style; even in the case $\kappa = \omega_1$ there is no form of properness available.

2. Any stationary subset of κ that arises during the iteration remains forever stationary.

Again we need to build a strong master condition for each model in a sequence of models built by taking successive Skolem hulls, but now the sequence has arbitrary successor length less than κ.
A $\square_\lambda$ sequence is used to thin out such a sequence to a subsequence of length at most $\lambda + 1$.

Δ_1 Definability of the Nonstationary Ideal: Main Result

3. A canonical stationary set $S_i \subseteq \kappa^+$ remains stationary unless in the course of the iteration its stationarity is explicitly killed in order to code some stationary $S \subseteq \kappa$.

Of course here we use the fact that the forcings to kill stationarity of selected S_i's (the "upper part") are κ-closed and the localisation forcings (the "lower part") are κ^+-cc.

3 implies that κ^+ is preserved.
As the entire iteration has a dense subset of size κ^+ all cardinals are preserved and GCH holds at cardinals $\geq \kappa$.
GCH holds below κ as no bounded subsets of κ are added.
Finally, by localisation together with the fact that no S_i "accidentally" loses its stationarity, we have that $S \subseteq \kappa$ is stationary iff S has a local witness, a Σ_1 property with parameter κ.
So the Theorem is proved.

Δ_1 Definability of the Nonstationary Ideal: Further Remarks

Descriptive Set Theory on κ-Baire Space

In classical Baire Space, the Baire Property for all Δ_1 $(= \Delta^1_2)$ sets of reals is equivalent to the existence of a Cohen real over $L[x]$ for each real x.

In our model where NS_κ is Δ_1 (for a successor κ) we have the existence of a κ-Cohen set over $L[x]$ for each $x \subseteq \kappa$.
As Halko-Shelah showed that NS_κ does not have the Baire Property, our result shows that the classical characterisation of the Δ_1 Baire Property does not generalise to successor κ.

Δ_1 Definability of the Nonstationary Ideal: Further Remarks

When $\kappa = \omega_1$

Wu and I showed that NS_{ω_1} can be both precipitous and Δ_1, starting with a measurable, extending a result of Magidor.
Woodin showed that NS_{ω_1} can be ω_1-dense, and therefore both Δ_1 and saturated, using ω Woodin cardinals.
Hoffelner and I get that NS_{ω_1} can be saturated and Δ_1 (together with a Σ^1_4 wellorder of ther reals) using just one Woodin cardinal.

There are many further questions to ask about the Δ_1 definability of NS_κ, regarding inaccessible κ, failures of GCH and saturation for $\kappa > \omega_1$, but I'll stop here.

THANKS!