Forcing with finite conditions

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Abstract

We present a generalisation to ω_2 of Baumgartner's forcing for adding a CUB subset of ω_1 with finite conditions.

The following well-known result appears in Baumgartner, Harrington, Kleinberg [2]. For the reader's convenience we provide a proof here.

Theorem 1 *Suppose that $X \subseteq \omega_1$. Then the following are equivalent:*

a. X contains a CUB subset in an outer model which preserves ω_1.
b. X is stationary.

Proof. (a) implies (b) because any two CUB sets must intersect. Conversely, consider the forcing P whose conditions are closed, countable subsets of X, ordered by end-extension. Clearly P adds a CUB subset to X; we must show that ω_1 is preserved.

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First a general comment about ω_1-preservation. We say that D is *predense below* p iff every condition below p is compatible with an element of D. Then ω_1-preservation is a consequence of the following:

$(*)$ For any p and D_i, $i < \omega$ with each D_i predense below p, there are $q \leq p$ and *countable* d_i, $i < \omega$ with $d_i \subseteq D_i$ and d_i predense below q for each $i < \omega$.

For if $(*)$ holds and p forces σ to be a function from ω to ω_1, then we can consider $D_i = \{q \mid \text{for some } \alpha < \omega_1,\ q \text{ forces } \sigma(i) = \alpha\}$; by $(*)$, there is $q \leq p$ and a countable β such that q forces $\sigma(i) < \beta$ for each $i < \omega$, and therefore q forces that σ is bounded.

Now to see that P preserves ω_1, suppose that $\langle D_i \mid i < \omega\rangle$ is a sequence of sets which are predense below p and choose a continuous elementary chain $\langle M_j \mid j < \omega_1\rangle$ of countable elementary submodels of H_θ, θ large, so that $X, p, \langle D_i \mid i < \omega\rangle$ belong to M_0 and $M_j \in M_{j+1}$ for each j. As $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is CUB, we can choose j so that $\alpha = M_j \cap \omega_1 \in X$. Then as each $D_i \cap M_j$ is predense below p on $P \cap M_j$, we can choose $p = p_0 \geq p_1 \geq \ldots$ so that p_{i+1} belongs to M_j and extends an element r_i of D_i for each $i < \omega$, and in addition $\bigcup_i p_i$ has supremum α. Then $q = \bigcup_i p_i \cup \{\alpha\}$ is a condition extending p, and for each i, $\{r_i\} \subseteq D_i$ is predense below q, proving $(*)$. $\square$

Next we ask the following.

Question. Suppose that X is a stationary subset of ω_1. Then is there a *cardinal-preserving* forcing P which adds a CUB subset to X?

The difficulty with the forcing used to prove Theorem 1 is that it will collapse $2^{\aleph_0}$ to ω_1, and therefore not preserve cardinals if CH fails. However, Baumgartner found a way of adding CUB sets with "finite conditions" which yields a positive answer to the above question (see [1]).

Theorem 2 *Let X be a stationary subset of ω_1. Then there is a forcing P which adds a CUB subset to X which preserves cofinalities.*

Proof. We use Uri Avraham's variant of Baumgartner's original technique (see [1]). A condition is a finite set p of disjoint closed intervals in ω_1 whose left endpoints belong to X. (We allow the one-point intervals $[\alpha, \alpha]$, $\alpha \in X$.) A condition q extends p iff q contains p.

It is clear that for generic G, $C_G =$ the set of all left endpoints of intervals in $\cup G$ is an unbounded subset of X. Each countable ordinal either belongs to some interval in G or fails to be a limit point of X; it follows that C_G is closed. As there are only ω_1 conditions, it only remains to show that ω_1 is preserved.

Suppose that p is a condition and D_i, $i < \omega$ are predense below p. Choose a continuous elementary chain $\langle M_j \mid j < \omega_1 \rangle$ of countable elementary submodels of H_θ, θ large, so that $X, p, \langle D_i \mid i < \omega \rangle$ belong to M_0 and $M_j \in M_{j+1}$ for each j. As $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is CUB, we can choose j so that $\alpha = M_j \cap \omega_1 \in X$. Let q be the condition $p \cup \{[\alpha, \alpha]\}$. If r extends q and $r_0 = r \restriction \alpha$ then every extension s_0 of r_0 in $P \cap M_j$ is compatible with r. This is because $[\alpha, \alpha]$ belongs to q. It follows that $d_i = D_i \cap M_j$ is predense below q for each i, as if r extends q then we can choose $s_0 \leq r_0$ which extends a condition in d_i, and therefore since s_0 is compatible with r, r is compatible with an element of d_i. Hence ω_1 is preserved. □

Now we look at the situation for ω_2. Unfortunately there is no analogue for Theorem 1.

Theorem 3 *(See [3].) Suppose that* $0^\#$ *exists. Then*

$\{X \subseteq \omega_2^L \mid X \in L$ *and* X *has a CUB subset in an inner model where* $\omega_2 = \omega_2^L\}$

is not constructible, and indeed has L-degree $0^\#$*. In particular, there are* X *which belong to the above set but have no CUB subset in any set-generic extension of* L *in which* $\omega_2 = \omega_2^L$*.*

However (under CH) there is a nice sufficient condition for a subset of ω_2 to contain a CUB in an extension preserving ω_1 and ω_2: $X \subseteq \omega_2$ is *fat stationary* iff $X \cap \mathrm{cof}\, \omega_1$ is stationary and for all α in $X \cap \mathrm{cof}\, \omega_1$, $X \cap \alpha$ contains a CUB subset of α.

Theorem 4 *Assume CH. If* $X \subseteq \omega_2$ *is fat stationary then there is a set-forcing extension preserving both* ω_1 *and* ω_2 *in which* X *contains a CUB subset.*

Proof. In analogy with the proof of Theorem 1, force with closed subsets of X of ordertype less than ω_2, ordered by end-extension. Countably closed models of size ω_1 and the fat stationarity of X are used as in the proof of Theorem 1 to show that if p is a condition and D_i, $i < \omega_1$, are predense below p then there is $q \leq p$ extending an element of D_i for each i. It follows that no new ω_1-sequences are added by the forcing and therefore both ω_1 and ω_2 are preserved. □

The forcing of Theorem 4 will collapse cardinals if GCH fails at ω_1. Avraham discovered a way of avoiding this problem, but still assuming CH. Is there a version for ω_2 of Baumgartner's forcing (as modified by Avraham) to add a CUB subset of a fat stationary set using finite conditions, without collapsing cardinals and without assuming CH? The following result provides a positive answer under the assumption of the existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ (an assumption weaker than CH).

Definition. $P_{\omega_1}(\omega_2)$ denotes the collection of countable subsets of ω_2. A subset S of $P_{\omega_1}(\omega_2)$ is *thin* iff for each $\alpha < \omega_2$, the set $\{x \cap \alpha \mid x \in S\}$ has cardinality at most ω_1.

Theorem 5 *Assume that there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$ and that $D \subseteq \omega_2$ is fat stationary. Then there is a forcing P that preserves cofinalities and adds a CUB subset of D.*

Remark. Thin *cofinal* subsets of $P_{\omega_1}(\omega_2)$ exist provably in ZFC. The existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ is strictly weaker than that of a special Aronszajn tree on ω_2. John Krueger has shown that thin stationary subsets of $P_{\omega_1}(\omega_2)$ do not exist if Martin's Maximum (MM) holds.

Proof of Theorem 5. Let D_1 denote $D \cap \operatorname{cof} \omega_1$. We can assume that D consists exclusively of limit ordinals and that $\alpha + \omega$ belongs to D whenever α belongs to D. Let T be a thin stationary subset of $P_{\omega_1}(\omega_2)$ and assume that T is closed under initial segments. Choose $B \subseteq \omega_2$ such that $T \subseteq L[B]$ and ω_2 equals (ω_2 of $L[B]$). An ordinal α is *good* iff it is a limit ordinal between ω_1 and ω_2 and for every $\beta < \alpha$, $\operatorname{cof} \beta$ equals ($\operatorname{cof} \beta$ in $L_\alpha[B]$). The set of good ordinals forms a CUB subset of ω_2.

For an ordinal α and a set x with $\alpha < \sup(x \cap \mathrm{Ord})$, let α_x denote the least ordinal $\geq \alpha$ in x. Note that if $\alpha < \alpha_x$ and x is Σ_1 elementary in some $L_\beta[B]$, β good, then α_x must have uncountable cofinality.

A condition is a pair $p = (A, S)$, where:

1. A is a finite set of disjoint closed intervals whose left endpoints belong to D. (We allow the one-point intervals $[\alpha, \alpha]$, $\alpha \in D$.) Let L_A denote the set of left endpoints of intervals in A.
2. S is a finite set of countable Σ_1 elementary submodels x of some $L_\beta[B]$, β good, such that $x \cap \mathrm{Ord}$ belongs to T and $\sup(x \cap \alpha)$ belongs to D whenever α belongs to $(x \cap D_1) \cup \{\omega_2\}$.
3. For each interval $I = [\alpha, \beta]$ in A and each $x \in S$:
   3a. If I intersects x then I belongs to x.
   3b. If $I = [\alpha, \beta]$ does not intersect x and $\alpha < \sup(x \cap \mathrm{Ord})$ then α_x belongs to L_A.
4. Let F_A be the set of all elements of L_A of cofinality ω_1, together with ω_2. For $x \in S$, the *F_A-height of x* is the least element of F_A greater than $\sup(x \cap \mathrm{Ord})$.
   4a. If x belongs to S and α belongs to F_A then $x \cap L_\alpha[B]$ belongs to S.
   4b. Suppose that $x, y \in S$ have the same F_A-height. Then $x \in y$, $y \in x$ or $x = y$.

Write $p = (A_p, S_p)$ and let L_p, F_p denote the L_A, F_A of 1, 4 above. q extends p iff A_q contains A_p and S_q contains S_p. For any condition q^* and $\alpha < \omega_2$ we define $q^* \restriction \alpha$ to be the pair $q = (A_q, S_q)$ where:

A_q is $A_{q^*} \cap L_\alpha[B]$,
S_q is $S_{q^*} \cap L_\alpha[B]$.

Claim 1. Suppose that p belongs to P.
(a) If C is a CUB subset of ω_2 then there exists $\alpha \in D_1 \cap C$ such that p belongs to $L_\alpha[B]$ and every subset of α in T belongs to $L_\alpha[B]$. For such α, obtain p^* from p by adding the interval $[\alpha, \alpha]$ to A_p (and otherwise leaving p unchanged). Then p^* is a condition extending p.
(b) Let α and p^* be as in part (a) and suppose that q^* extends p^*. Then $q^* \restriction \alpha = q$ is a condition in $L_\alpha[B]$ extending p such that every extension of q in $L_\alpha[B]$ is compatible with q^*.

Proof of Claim 1:

(a) Such α exist since D_1 is stationary and $T \subseteq L_{\omega_2}[B]$ is thin. Property 1 is satisfied by p^* as α is greater than the right endpoint of any interval in A_p. Property 2 is the same for p^* as for p. Property 3a is the same for p^* as for p, as α does not belong to any element of S_p. Property 3b is the same for p^* as for p, as α is greater than $\sup(x \cap \mathrm{Ord})$ for any $x \in S_p$. And property 4 holds for p^* as $F_{p^*} = F_p \cup \{\alpha\}$, $x \cap L_\alpha[B] = x$ for all $x \in S_p$ and x, $y \in S_p$ have the same F_{p^*}-height iff they have the same F_p-height.

is the same for p^* as for p, as α does not belong to any element of S_p.

(b) Suppose that q^* extends p^* and set $q = q^* \restriction \alpha$.

Subclaim 1. q is a condition in $L_\alpha[B]$ which extends p.

Clearly q, if a condition, extends p since q^* does and p belongs to $L_\alpha[B]$. To verify that q is a condition, we need only verify properties 3b and 4.

3b. Assume that $I \cap x = \emptyset$ and the left endpoint β of $I = [\beta, \gamma]$ is less than $\sup(x \cap \mathrm{Ord})$, where I belongs to $A_{q^*} \cap L_\alpha[B]$ and x belongs to $S_{q^*} \cap L_\alpha[B]$. Then since q^* is a condition, β_x is the left endpoint of some interval J in S_{q^*}. But since $[\alpha, \alpha]$ belongs to A_{q^*}, the right endpoint of J is less than α and therefore J belongs to $A_{q^*} \cap L_\alpha[B] = A_q$.

For property 4, first note that $F_q = F_{q^*} \cap \alpha$, together with ω_2.

4a. If x is in S_q and $\beta \in F_q$ then $x \cap L_\beta[B]$ is in S_{q^*} and therefore also in $S_q = S_{q^*} \cap L_\alpha[B]$, since, using our hypothesis on α, $x \cap L_\beta[B]$ is an element of $L_\alpha[B]$.

4b. If $x, y \in S_q$ have the same F_q-height then since they both belong to $L_\alpha[B]$, they have the same F_{q^*}-height. Thus the desired conclusion follows as $x, y \in S_{q^*}$ and q^* is a condition.

Now suppose that r is an extension of q, and r belongs to $L_\alpha[B]$. We must find a common extension of r and q^*. We define t by

$A_t = A_r \cup A_{q^*}$,
$S_t = S_r \cup S_{q^*}$.

Subclaim 2. t is a condition extending both r and q^*.

Clearly t, if a condition, extends both r and q^*. We now verify that t is a condition, by verifying properties 1-4.

1. The intervals in A_t are disjoint, as r is a condition extending q, all intervals in A_r have right endpoint less than α and all intervals in A_{q^*} not in A_q have left endpoint at least α.

2. Clear.

3. Suppose that I is an interval in $A_t - A_r$ and x belongs to S_r. Then $\sup(x \cap \text{Ord})$ is less than α and the left endpoint of I is at least α. So property 3 is vacuous in this case. Suppose that I belongs to A_r and x belongs to $S_t - S_r$. Then $x \cap L_\alpha[B]$ belongs to $S_q \subseteq S_r$ and therefore property 3 holds for I and $x \cap L_\alpha[B]$. It follows that 3a holds for I and x, since if I intersects x it must also intersect $x \cap L_\alpha[B]$. And 3b holds for I and x: If I is disjoint from x and the left endpoint β of I is less than $\sup(x \cap \text{Ord})$ then I is also disjoint from $x \cap L_\alpha[B]$ and either (i) β is less than $\sup(x \cap \alpha)$, in which case $\beta_x = \beta_{x \cap \alpha}$ and therefore the result follows since r is a condition, (ii) $\beta_x = \alpha$, in which case the result follows since $[\alpha, \alpha]$ belongs to A_{q^*}, or (iii) $\beta_x = \alpha_x$, in which case the result follows since q^* is a condition. The remaining cases, where I belongs to A_r and x belongs to S_r, or where I belongs to $A_t - A_r$ and x belongs to $S_t - S_r$, are immediate since r and q^* are conditions.

4a. We must show that if x belongs to S_t and $\beta \in F_t$ then $x \cap L_\beta[B]$ belongs to S_t. If x belongs to S_r then either β is in F_r, in which case $x \cap L_\beta[B]$ belongs to $S_r \subseteq S_t$ since r is a condition, or β is at least α, in which case $x \cap L_\beta[B] = x \in S_r \subseteq S_t$. If x belongs to S_{q^*} then either β is in F_{q^*}, in which case the result follows since q^* is a condition, or β is in F_r, in which case $x \cap L_\beta[B] = (x \cap L_\alpha[B]) \cap L_\beta[B] \in S_r \subseteq S_t$, since $x \cap L_\alpha[B] \in S_q \subseteq S_r$ and r is a condition.

4b. We must show that if $x, y \in S_t$ have the same F_t-height, then $x \in y, y \in x$ or $x = y$. If x belongs to S_r then the F_t height of x is at most α and therefore y also belongs to S_r; thus x, y have the same F_r-height and the result follows since r is a condition. If x belongs to $S_{q^*} - S_r$ then the F_t-height of x is greater than α, and therefore y also belongs to S_{q^*}; thus x, y have the same F_{q^*}-height and the desired conclusion follows since q^* is a condition.

This completes the proof of Claim 1.

Claim 2. Suppose that p belongs to P.

(a) For any CUB $C \subseteq P_{\omega_1}(\omega_2)$ there exists a countable elementary submodel

x of $L_{\omega_2}[B]$ such that $x \cap \mathrm{Ord}$ belongs to $C \cap T$, p belongs to x and whenever α belongs to $(x \cap D_1) \cup \{\omega_2\}$, $\sup(x \cap \alpha)$ belongs to D. For such x obtain p^* from p by adding $x \cap L_\alpha[B]$ to S_p for all $\alpha \in F_p$ (and otherwise leaving p unchanged). Then p^* is a condition extending p.
(b) Let x and p^* be as in part (a). Then if q^* extends p^* there is q in x extending p such that every extension of q in x is compatible with q^*.

Proof of Claim 2:

(a) To see that such x exist, argue as follows. Choose β in D_1 such that $C \cap P_{\omega_1}(\beta)$ is CUB in $P_{\omega_1}(\beta)$. Also choose $y \in T$ such that $y \cap \beta$ belongs to $C \cap P_{\omega_1}(\beta)$ and $\sup(y \cap \alpha)$ belongs to D whenever α belongs to $(y \cap \beta \cap D_1) \cup \{\beta\}$. As T is closed under initial segments, $x = y \cap \beta$ belongs to T and has the desired properties.

Clearly p^*, if a condition, extends p. To verify that p^* is a condition we need only check properties 3 and 4.
3a. As p belongs to x, each interval in A_p belongs to x and therefore the conclusion of 3a holds for x. It follows easily that 3a also holds for $x \cap L_\alpha[B]$ whenever α belongs to F_p. 3a holds for other elements of S_{p^*} since p is a condition.
3b. This is vacuous for $x \cap L_\alpha[B]$, $\alpha \in F_p$, and holds for other elements of S_{p^*} since p is a condition.
4a. This is true for $x \cap L_\alpha[B]$, $\alpha \in F_p$, by definition of p^*, and for other elements of S_{p^*} since p is a condition.
4b. Suppose that $y, z \in S_{p^*}$ have the same F_{p^*}-height ($= F_p$-height). If both y, z belong to S_p then the desired conclusion follows since p is a condition. Assume that $y = x \cap L_\alpha[B]$ where α belongs to F_p. If z belongs to S_p then z belongs to x and since it has the same F_p-height as y, must also belong to $L_\alpha[B]$; hence z belongs to y. If z is of the form $x \cap L_\beta[B]$, $\beta \in F_p$, and has the same F_p-height as y then $z = y$, since the F_p-height of $x \cap L_\beta[B]$ equals β for any $\beta \in F_p$.

(b) Let q^* extend p^* and define q as follows:

A_q is $A_{q^*} \cap x$,
S_q is $S_{q^*} \cap x$

Subclaim 1. q is a condition in x extending p.

Clearly q, if a condition, extends p since q^* extends $p^* \leq p$ and p belongs to x. To check that q is a condition we need only verify properties 3b and 4.
3b. Suppose that I belongs to A_q, I is disjoint from y and the left endpoint α of I is less than $\sup(y \cap \mathrm{Ord})$, where y belongs to S_q. Then α_y is the left endpoint of some $J \in A_{q^*}$ since q^* is a condition. Since J intersects y, J must belong to y and therefore also to x, since y belongs to x. Thus J belongs to A_q.
4a. If y belongs to S_q and α belongs to $F_q \cap \omega_2$ then $y \cap L_\alpha[B]$ belongs to S_{q^*} since q^* is a condition. Since both y and α belong to x, we get $y \cap L_\alpha[B] \in S_q$. If y belongs to S_q then $y \cap L_{\omega_2}[B] = y$ and therefore belongs to S_q.
4b. Suppose that $y \in S_q$ has F_q-height α and F_{q^*}-height β. Suppose that β is less than $\sup(x \cap \mathrm{Ord})$. Then either β equals α or is the left endpoint of some interval in A_{q^*} disjoint from x. In the latter case, β_x is the left endpoint of some interval in $A_{q^*} \cap x = A_q$ and therefore β_x belongs to F_q, since it must have uncountable cofinality. Thus $\beta_x = \alpha$. So we conclude that the F_{q^*}-height of y is the least $\beta \in F_{q^*}$ such that either β is less than $\sup(x \cap \mathrm{Ord})$ and $\beta_x = \alpha$, or β is greater than $\sup(x \cap \mathrm{Ord})$. Therefore the F_{q^*}-height of $y \in S_q$ is uniquely determined by the F_q-height of y. If $y, z \in S_q$ have the same F_q-height then they therefore also have the same F_{q^*}-height, and since q^* is a condition, either $y \in z$, $z \in y$ or $y = z$, as desired.

Now suppose that r in x extends q. We must find a common extension t of r and q^*. We define t by:

$A_t = A_r \cup A_{q^*}$
$S_t = S_r \cup \{y \cap L_\alpha[B] \mid y \in S_{q^*},\ \alpha \in F_r\}$.

Subclaim 2. t is a condition extending both r and q^*.

Clearly t, if a condition, extends both r and q^*. We show now that t is a condition.
1. Suppose that I is an interval in A_{q^*} but not in A_r. Then I is disjoint from x. If the left endpoint α of I is at least $\sup(x \cap \mathrm{Ord})$, then I is disjoint from all intervals in A_r, since the latter belong to x. Otherwise α_x is the left endpoint of some interval J in A_q. It follows that the intervals in A_r are disjoint from I, as they belong to x and are either equal to or disjoint from J. Thus A_t consists of pairwise disjoint intervals.
2. We must show that if y belongs to S_t and $\alpha \in (y \cap D_1) \cup \{\omega_2\}$ then $\sup(y \cap \alpha)$

belongs to D. This is clear if y belongs to S_r since r is a condition. It also holds if y belongs to S_{q^*} since q^* is a condition. This implies the result for arbitrary $y \in S_t$ when α is not ω_2. It remains to show: If $y \in S_{q^*}$, $\alpha \in F_r$ then $\sup(y \cap \alpha)$ belongs to D. Let β be least in $F_{q^*} - \alpha$. If β is not the F_{q^*}-height of $y \cap L_\beta[B]$ then $y \cap \alpha = y \cap \beta$ and therefore $\sup(y \cap \alpha)$ belongs to D since q^* is a condition. Otherwise, $y \cap L_\beta[B]$ and $x \cap L_\beta[B]$ have the same F_{q^*}-height, since α belongs to x. Since q^* is a condition, either $y \cap L_\beta[B] \in x \cap L_\beta[B]$, $x \cap L_\beta[B] \in y \cap L_\beta[B]$ or $y \cap L_\beta[B] = x \cap L_\beta[B]$. In the first case, $y \cap L_\beta[B]$ belongs to S_r so $y \cap \alpha = (y \cap \beta) \cap \alpha$ belongs to D since r is a condition. In the latter two cases, α belongs to $y \cap D_1$, and therefore the result follows since q^* is a condition.

3a. Suppose that I is an interval in A_t, y belongs to S_t and I intersects y. We must show that I belongs to y. First we consider the case where I belongs to A_r and y belongs to $S_t - S_r$. Write $I = [\alpha, \beta]$ and $y = z \cap L_\gamma[B]$, where z belongs to $S_{q^*} - S_r$ and γ belongs to F_r. Let β^* be the least element of F_{q^*} greater than α. Since we have shown that A_t consists of pairwise disjoint intervals, it follows that β^* is greater than β. Therefore the F_{q^*}-heights of $x \cap L_{\beta^*}[B]$ and $z \cap L_{\beta^*}[B]$ are both β^*, the former since β belongs to x and the latter since z intersects $I = [\alpha, \beta]$. Thus either $z \cap L_{\beta^*}[B] \in x \cap L_{\beta^*}[B]$, $x \cap L_{\beta^*}[B] \in z \cap L_{\beta^*}[B]$ or $x \cap L_{\beta^*}[B] = z \cap L_{\beta^*}[B]$. The first possibility implies that I intersects $y \cap L_{\beta^*}[B] = (z \cap L_{\beta^*}[B]) \cap L_\gamma[B] \in S_r$, and therefore since r is a condition, I belongs to $y \cap L_{\beta^*}[B] \subseteq y$, as desired. The second and third possibilities imply that y contains $x \cap L_{\beta^*}[B]$ as a subset and therefore I as an element. Now consider the case where I belongs to $A_t - A_r$ and y belongs to S_r. Then I belongs to A_{q^*} and intersects x, which belongs to S_{q^*}. Thus I belongs to x, contradicting the fact that I does not belong to $A_q \subseteq A_r$. The case where I belongs to A_r and y belongs to S_r follows since r is a condition. Finally, if I belongs to $A_t - A_r$ and y belongs to $S_t - S_r$, write $y = z \cap L_\alpha[B]$ where $z \in S_{q^*}$ and $\alpha \in F_r$. Since q^* is a condition, I belongs to z. If I does not belong to y, then I intersects x and therefore belongs to x, again since q^* is a condition. But this contradicts the hypothesis that I does not belong to A_r.

3b. Suppose that $I = [\alpha, \beta]$ belongs to A_t, y belongs to S_t, I is disjoint from y and α is less than $\sup(y \cap \mathrm{Ord})$. We must show that α_y is the left endpoint of some interval in A_t. First we consider the case where I belongs to A_r and y belongs to $S_t - S_r$. Write $y = z \cap L_\gamma[B]$ where z belongs to S_{q^*} and γ belongs to F_r. Let β^* be the least element of F_{q^*} greater than β. If $\alpha_y = \beta^*$

then α_y is the left endpoint of some interval in A_{q^*} and we are done. If $\alpha_y > \beta^*$, then let J be the interval of A_{q^*} with left endpoint β^*. Since q^* is a condition and $\alpha_y = \alpha_z$, J is not an element of z and therefore is disjoint from z. Since q^* is a condition, $\alpha_y = (\beta^*)_z$ is the left endpoint of some interval of A_{q^*}, as desired. Finally, if $\beta < \alpha_y < \beta^*$, it follows that $x \cap L_{\beta^*}[B]$ and $z \cap L_{\beta^*}[B]$ both have F_{q^*}-height β^*, and therefore $x \cap L_{\beta^*}[B] \in z \cap L_{\beta^*}[B]$, $z \cap L_{\beta^*}[B] \in x \cap L_{\beta^*}[B]$ or $x \cap L_{\beta^*}[B] = z \cap L_{\beta^*}[B]$. The first and third of these possibilities contradict our hypothesis that $I \in x$ is disjoint from y. In the second possibility, $z \cap L_{\beta^*}[B]$ belongs to S_q and since $\alpha_y = \alpha_z$ is less than β^*, we have that α_y is the left endpoint of some interval in A_r since r is a condition. Next we consider the case where I belongs to $A_t - A_r$ and y belongs to S_r. Thus I belongs to A_{q^*} and must be disjoint from x, else it would belong to x and therefore to A_r. As α is less than $\sup(x \cap \mathrm{Ord})$, it follows that α_x is the left endpoint of some interval in A_{q^*}, which in fact belongs to A_r. If $\alpha_y = \alpha_x$ then we are done. Otherwise α_y equals $(\alpha_x)_y$, which must be the left endpoint of an interval in A_r, since r is a condition. The remaining two cases, where either I belongs to A_r and y belongs to S_r, or where I belongs to $A_t - A_r$ and y belongs to $S_t - S_r$, follow since both r and q^* are conditions.

4a. We must show that if y belongs to S_t and α belongs to F_t then $y \cap L_\alpha[B]$ belongs to S_t. This is clear if y belongs to S_r and α belongs to F_r, or if y belongs to F_{q^*} and α belongs to F_{q^*}, since r and q^* are conditions. This is also true if y belongs to S_{q^*} and α belongs to F_r, by definition of S_t. And we may assume that y belongs to $S_r \cup S_{q^*}$. So we need only check the case where y belongs to S_r, α belongs to F_{q^*} and α is less than $\sup(y \cap \mathrm{Ord})$. If α belongs to x then it also belongs to F_r so we are done since r is a condition. Otherwise α_x is defined and belongs to F_r. So since r is a condition, $y \cap L_\alpha[B] = y \cap L_{\alpha_x}[B]$ belongs to S_r.

4b. We must show that if $y, z \in S_t$ have the same F_t-height then either $y \in z$, $z \in y$ or $y = z$. Note that y, z also have the same F_r-height and the same F_{q^*}-height. If y, z both belong to S_r then the desired conclusion follows since r is a condition. Suppose that y, z are of the form $y^* \cap L_\alpha[B]$, $z^* \cap L_\beta[B]$, respectively, where y^*, z^* belong to S_{q^*} and $\alpha, \beta \in F_r$. We may assume that α, β are the F_r-heights of y, z, respectively, and therefore $\alpha = \beta$. Let α^* be the common F_{q^*}-height of y, z. Then $y^* \cap L_{\alpha^*}[B]$, $z^* \cap L_{\alpha^*}[B]$ have F_{q^*}-height α^* and therefore since q^* is a condition, we have $y^* \cap L_{\alpha^*}[B] \in z^* \cap L_{\alpha^*}[B]$, $y^* \cap L_{\alpha^*}[B] = z^* \cap L_{\alpha^*}[B]$ or $z^* \cap L_{\alpha^*}[B] \in y^* \cap L_{\alpha^*}[B]$. The second possibility yields $y = z$. The first possibiilty implies that y belongs to $z^* \cap L_{\alpha^*}[B]$ since it is an initial segment of $y^* \cap L_{\alpha^*}[B]$; as $y \in L_\alpha[B]$ we get $y \in z^* \cap L_\alpha[B] = z$. The third possiblity is handled identically to the first, with y and z switched. Finally assume that y belongs to S_r and $z = z^* \cap L_\alpha[B]$ where $z^* \in S_{q^*}$ and $\alpha \in F_r$. We may assume that α is the F_r-height of z, which is also the F_r-height of y. Let α^* be the common F_{q^*}-height of y and z. Then α^* is also the F_{q^*}-height of $x \cap L_{\alpha^*}[B]$, since x contains y, and is the F_{q^*}-height of $z^* \cap L_{\alpha^*}[B]$. Since q^* is a condition, we have $z^* \cap L_{\alpha^*}[B] \in x \cap L_{\alpha^*}[B]$, $x \cap L_{\alpha^*}[B] \in z^* \cap L_{\alpha^*}[B]$ or $z^* \cap L_{\alpha^*}[B] = x \cap L_{\alpha^*}[B]$. Under the first possibility, $(z^* \cap L_{\alpha^*}[B]) \cap L_\alpha[B] = z$ belongs to S_r, so we are done since r is a condition. The second and third possibilities imply that $y \in z^* \cap L_\alpha[B] = z$.

This completes the proof of Claim 2.

Claim 1 implies that ω_2 is preserved. Claim 2 implies that ω_1 is preserved. As P has cardinality ω_2, all cofinalities are preserved.

Claim 3. Let G be P-generic and $C_G = \{\gamma \mid \gamma$ is a left endpoint of some interval in $\cup\{A_p \mid p \in G\}\}$. Then C_G is a CUB subset of D.

Proof of Claim 3:

It follows from *Claim 1 (a)* that C_G is unbounded. We show that C_G is closed.

Suppose that p is a condition and for the sake of contradiction, $p \Vdash (\alpha \in \text{Lim } C_G$ and $\alpha \notin C_G)$. We may assume that for each $y \in S_p$, if α_y is defined and forced by some extension of p to belong to C_G, then α_y is the left endpoint of some interval in A_p; otherwise we can enlarge A_p without changing S_p to guarantee this property. Note that for $q \leq p$, α does not belong to any interval in A_q, else q forces either that α belongs to C_G or is not the limit of elements of C_G.

Suppose that y belongs to S_p, α is not in Lim $(y \cap \text{Ord})$ but α is less than $\sup(y \cap \text{Ord})$. Then observe that α_y must be a left endpoint of some interval in A_p, else by requirement 3b on conditions, no extension of p can introduce a new interval with left endpoint between $\sup(y \cap \alpha)$ and α, and hence p cannot force that α is a limit point of C_G. Let β be the least element of F_p

greater than α and consider $S = \{y \in S_p \mid \alpha \leq \sup(y \cap \mathrm{Ord}) < \beta\}$. Then by requirement 4b on conditions, the elements of S form an $\in$-chain.

Assume first that $y \cap \alpha$ is cofinal in α for some $y \in S$, and let y_0 be the $\in$-least such. Note that if α_{y_0} is defined and is the left endpoint of some interval in A_p then α must belong to D, by requirement 2 on conditions. We show that we can extend p to force either that α belongs to C_G or that α is not a limit point of C_G, achieving the desired contradiction. Note that $D \cap y_0 \cap \alpha$ must be cofinal in α, as there are cofinally many $\gamma < \alpha$ which are forced by extensions of p into C_G and for any such $\gamma \notin y_0$, γ_{y_0} belongs to D by requirement 3b on conditions. Since $\gamma + \omega$ belongs to D whenever γ does, it follows that $D \cap y_0 \cap \alpha \cap \mathrm{cof}\ \omega$ is also cofinal in α.

If α_{y_0} is defined and not the left endpoint of some interval in A_p, then let γ be an element of $D \cap y_0 \cap \alpha \cap \mathrm{cof}\ \omega$ greater than the right endpoint of any interval of A_p with left endpoint less than α, and larger than $\sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$. We claim a condition results when the interval $I = [\gamma, \alpha_{y_0}]$ is added to p: I is disjoint from intervals of A_p with left endpoint less than α by choice of γ. And it is disjoint from intervals of A_p with left endpoint greater than α since by assumption, α_{y_0} is not the left endpoint of an interval of A_p, and therefore by 3a, 3b neither is any ordinal between α and α_{y_0}. I does not intersect any $y \in S_p$ with $\sup(y \cap \beta) < \alpha$ by choice of γ. I does not intersect any $y \in S_p$ with $\sup(y \cap \alpha) < \alpha < \sup(y \cap \beta)$: For such y we have $y \cap \beta \in y_0$ and therefore $y \cap \beta \subseteq y_0$; also $\alpha_y > \alpha_{y_0}$ since, as observed earlier, α_y must be a left endpoint of some interval of A_p and by assumption α_{y_0} is not. Any other $y \in S_p$ contains y_0 as an element and therefore as a subset, and therefore also the interval I as an element. For those $y \in S_p$ disjoint from I with $\gamma < \sup(y \cap \mathrm{Ord})$, we have $\gamma_y = \alpha_y$, and as observed earlier, α_y is the left endpoint of an interval of A_p. This completes the verification that adding I to p results in a condition.

If α_{y_0} is defined and the left endpoint of some interval in A_p, then let $I = [\alpha, \alpha]$. We claim that a condition results when we add I to p: Of course I is disjoint from all intervals of A_p since α does not belong to any such interval. Trivially, if I intersects $y \in S_p$ then it belongs to y. If I is disjoint from $y \in S_p$ and $\alpha < \sup(y \cap \mathrm{Ord})$, then $\alpha_y \geq \alpha_{y_0}$, as otherwise $y \cap L_\beta[B] \in S$, $y_0 \in y \cap L_\beta[B]$ and therefore $\alpha = \sup(y_0 \cap \alpha_{y_0}) \in y$, against our hypothesis.

So α_y must be a left endpoint of some interval of A_p, else by requirements 3a,3b on conditions, α_{y_0} could not be. This completes the verification that adding I to p results in a condition.

If α_{y_0} is undefined then again set $I = [\alpha, \alpha]$. We claim that a condition results when we add I to p: By the argument of the previous paragraph, it suffices to show that if I is disjoint from $y \in S_p$ and $\alpha < \sup(y \cap \mathrm{Ord})$ then α_y is the left endpoint of some interval of A_p. If α_y is at least β, then this must be the case, as otherwise β could not be the left endpoint of such an interval. If α_y is less than β then $y \cap L_\beta[B]$ belongs to S and therefore either y_0 is an element of y, contradicting $\alpha \notin y$, or $\alpha_y \geq \alpha_{y_0}$. If α_y equals α_{y_0} then α_y is the left endpoint of some interval of A_p by hypothesis, and if α_y is greater than α_{y_0} then it must be the left endpoint of an interval of A_p, else α_{y_0} could not be the left endpoint of such an interval. This completes the verification that adding I to p results in a condition.

Lastly, we treat the case where $y \cap \alpha$ is not cofinal in α for all $y \in S$. In this case we choose $I = [\gamma, \alpha]$, where γ is an element of $D \cap \alpha \cap \mathrm{cof}\, \omega$ greater than the right endpoint of any interval of A_p with left endpoint less than α, and larger than $\sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$. We claim that a condition results when we add I to p: I is disjoint from all intervals in A_p by choice of γ. I is disjoint from each $y \in S_p$, as $y \cap \alpha$ is contained in γ by choice of γ and the case hypothesis, and if α belongs to y, we have $\alpha < \sup(y \cap \mathrm{Ord})$, which, as observed earlier, implies that $\alpha_y = \alpha$ the left endpoint of an interval of A_p, an impossibility. If y belongs to S_p and $\gamma < \sup(y \cap \mathrm{Ord})$ then α_y must be the left endpoint of an interval of A_p, as observed earlier. This completes the verification that adding I to p results in a condition.

This completes the proof of Theorem 5.

A remark and some questions. The hypothesis that D is fat stationary is not necessary for Theorem 5. The proof only uses that there is a thin stationary subset S of $P_{\omega_1}(\omega_2)$ such that for $x \in S$, $\sup(x \cap \alpha)$ belongs to D whenever α belongs to $x \cap D \cap \mathrm{Lim}\, D$ or $\alpha = \omega_2$. However this hypothesis is not substantially weaker than the one stated in Theorem 5 as, at least under CH, there is a countably distributive, cofinality-preserving forcing that adds a fat stationary subset to such a set D. Can the assumption of a thin stationary subset of $P_{\omega_1}(\omega_2)$ be entirely eliminated from the statement of Theorem 5? Is Theorem 5 still true if CH is added to both its hypothesis and conclusion? And is there a version of Theorem 5 for ω_3?

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