An Elementary Approach to the Fine Structure of L

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We present here an approach to the fine structure of L based solely on elementary modeltheoretic ideas, and illustrate its use in a proof of Global Square in L. We thereby avoid the Lévy hierarchy of formulas and the subtleties of master codes and projecta, introduced by Jensen [1972] in the original form of the theory. Our theory could appropriately be called "Hyperfine Structure Theory", as we make use of a hierarchy of structures and hull operations which refines the traditional L_α– or J_α–sequences with their Σ_n-hull operations.

1 Introduction

In 1938, K. Gödel defined the model L of set theory to show the relative consistency of Cantor's Continuum Hypothesis. L is defined as a union

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$$

of initial segments which satisfy: $L_0 = \emptyset$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit ordinals λ, and, crucially, $L_{\alpha+1}$ = the collection of 1st order definable subsets of L_α. Since every transitive model of set theory must be closed under 1st order definability, L turns out to be the smallest inner model of set theory. Thus it occupies the central place in the set theoretic spectrum of models.

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The proof of the continuum hypothesis in L is based on the very uniform hierarchical definition of the L–hierarchy. The *Condensation Lemma* states that if $\pi : M \to L_\alpha$ is an elementary embedding, M transitive, then $M = L_{\overline{\alpha}}$ for some $\overline{\alpha}$; the lemma can be proved by induction on α. If a real, i.e., a subset of ω, is definable over some L_α, then by a Löwenheim-Skolem argument it is definable over some countable M as above, and hence over some $L_{\overline{\alpha}}$, $\overline{\alpha} < \omega_1$. This allows one to list the reals in L in length ω_1 and therefore proves the Continuum Hypothesis in L.

This type of argument has been refined in a striking way in R. Jensen's *Fine Structure Theory* [1972]. Roughly speaking, Jensen was able to find, uniformly, a Skolem function for Σ_n–formulae over L_α which itself has a Σ_n–definition over L_α. If an interesting phenomenon like the collapse or the singularisation of an ordinal is Σ_n–definable over L_α we can use the Σ_n–Skolem function to achieve that effect canonically. Simultaneously, the Σ_n–Skolem function produces substructures which condense down to $L_{\overline{\alpha}}$'s, preserving the definition of the Skolem function. So the construction over L_α will "cohere" nicely with an analogous construction over $L_{\overline{\alpha}}$ which is essential for the coherence properties in Jensen's principles $\square$ and "morass". These principles have proved to be central to the resolution of a number of important questions in set theory, not necessarily connected to the constructible universe.

The method of Jensen presents a veritable *tour de force* even by today's standards of set theoretical sophistication. The L_α's, or rather the J_α's, have to be expanded by (iterated) projecta, standard parameters, mastercodes and reducts to ensure the preservation of higher levels of the Lévy–hierarchy of formulae in condensation arguments. Only after understanding those fine-structural notions can one turn to the *combinatorial* aspects of a $\square$–proof, for example. These complications have motivated attempts to simplify fine structure theory. Silver and then Magidor [1990] work with Skolem functions for Σ_n–formulae which are not quite Σ_n–definable but are still preserved in condensations. Such "approximations" to fine structure theory were particularly successfull in mild applications of the theory as, e.g., in the proof of the famous Jensen Covering Theorem. Earlier, Silver had employed "machines" on ordinals which compute the truth predicate for the L_α–hierarchy and which allow to concentrate on the combinatorics of Jensen's constructions (Silver [197?], Devlin [1984] and Richardson [1978]). The approach of

Friedman [1997], based on Jensen's Σ^* approach, eliminates certain unnatural parameters, but is otherwise very close in spirit to Jensen's original fine structure theory.

In this article we present a natural alternative to fine structure theory, employing elementary concepts from model theory rather than ideas derived from recursion theory. The approach shares some technical properties with Silver machines but we are solely working on the basis of the familiar L_α–hierarchy which we shall expand by *restricted* Skolem functions.

As a motivation let us consider the process of singularisation of an ordinal β in L. Suppose $L \models \beta$ is singular. Let γ be minimal such that over L_γ we can define a cofinal subset C of β of smaller ordertype; we can assume that C takes the form

$$C = \{z \in \beta \mid \exists x < \alpha : z \text{ is } <_L - \text{minimal such that } L_\gamma \models \varphi(z, \vec{p}, x)\}$$

where $\alpha < \beta$, φ is a first order formula, $\vec{p}$ is a parameter sequence from L_γ. If

$$S_\varphi(\vec{y}, x) = \text{the } <_L -\text{minimal } z \text{ such that } \varphi(z, \vec{y}, x)$$

is the term for the Skolem function for φ, then

$$C = S_\varphi^{L_\gamma}{}''\{\vec{p}^\frown x \mid x < \alpha\}$$

and β is singularised by $S_\varphi^{L_\gamma}$ *restricted to arguments lexicographically smaller* than the tuple $\vec{p}^\frown\alpha$, where the lexicographical order $<^{\text{lex}}$ is derived from the $<_L$–order. The foregoing suggests saying that β is singularised *at the location* $(\gamma, \varphi, (\vec{p}, \alpha))$, and that the right singularising structure for β is of the form

$$L_{(\gamma,\varphi,\vec{p}^\frown a)} = (L_\gamma, \in, <_L, \ldots, S_{\varphi_0}^{L_\gamma}, S_{\varphi_1}^{L_\gamma}, \ldots, S_{\varphi_n}^{L_\gamma} \restriction \{\vec{w} \mid \vec{w} <^{\text{lex}} \vec{p}^\frown\alpha\}, \ldots);$$

where $\varphi_0, \varphi_1, \ldots$ is a fixed ω–enumeration of the $\in$–formulae, and where $\varphi_n = \varphi$. The inclusion of the Skolem functions for all subformulae of φ_n will ensure the condensation property for such singularising structures.

These structures provide us with a very fine interpolation between successive L_γ–levels:

$$L_\gamma, \ldots\ldots, L_{(\gamma,\varphi,\vec{p}^\frown\alpha)}, \ldots\ldots, L_{\gamma+1}, \ldots\ldots$$

The enriched hierarchy satisfies *Condensation* and a *Finiteness Property* which is reminiscent of the key property of Silver machines.

In the present article we apply the method to establish a Global Square principle in L, incorporating ideas of J. Silver (see Devlin [1984]) and S. Friedman [1997] into the proof. We have also found very natural arguments for $(\kappa, 1)$–morasses and for the Covering Theorem which we plan to publish in a subsequent article.

It is our hope that our approach will make the Fine Structure of L more accessible to a wide audience of set-theorists, and separate definability issues from the combinatorial content of Jensen's arguments.

2 Names and Locations

For any $\alpha \in \mathrm{ORD}$, $\varphi(u, \vec{v})$ a first–order formula with $n+1$ free variables, and $\vec{x}$ a sequence from L_α of length n, let $I(\alpha, \varphi, \vec{x})$ denote $\{y \in L_\alpha \mid L_\alpha \models \varphi(y, \vec{x})\}$. Thus we can think of the above triples $(\alpha, \varphi, \vec{x})$ as *names* for elements of L. A central idea in our theory is to also view $(\alpha, \varphi, \vec{x})$ as a *location* for the structure $L_{(\alpha,\varphi,\vec{x})}$ in the fine hierarchy with an associated *hull operation* $L_{(\alpha,\varphi,\vec{x})}\{\cdot\}$ which approximates the usual Skolem hull operation on subsets of L_α. Before we define these notions we first discuss the ordering of names (=locations) and prove a condensation result for "constructibly–closed" subsets of L_α.

Wellorder names and constructible sets in the standard way as follows: Consider $\in$–formulae built using $\neg$, $\wedge$, $\vee$ and the existential quantifier $\exists$. We agree that every formula φ has a distinguished variable used for the I–operation and for existential quantifications. When we write $\varphi(u, \vec{x})$, we intend that u is distinguished in φ; then $\exists u \varphi$ with any choice of distinguished variable is a new permitted formula. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an ω–ordering of permitted formulas, subformulas appearing earlier, which we assume to be fixed throughout this article.

We take $<_0$ to be the vacuous ordering on $L_0 = \emptyset$. If $<_\alpha$ is defined as a wellordering of L_α then order sequences from L_α by $\vec{x} <_\alpha^{\mathrm{lex}} \vec{y}$ iff $\vec{x}$ is lexicographically less then $\vec{y}$, using $<_\alpha$ on the components of $\vec{x}$ and $\vec{y}$. Names $(\beta, \varphi, \vec{x})$ where $\beta \leq \alpha$ are ordered by:

$$
\begin{aligned}
(\beta, \varphi_m, \vec{x}) \mathrel{\tilde{<}} (\gamma, \varphi_n, \vec{y}) \text{ iff } \quad & (\beta < \gamma) \\
& \vee\ (\beta = \gamma \wedge m < n) \\
& \vee\ (\beta = \gamma \wedge m = n \wedge \vec{x} <_\beta^{\mathrm{lex}} \vec{y}).
\end{aligned}
$$

And for $y \in L_{\alpha+1}$ let $N(y)$ denote the $\tilde{<}$–least $(\beta, \varphi, \vec{x})$ such that $I(\beta, \varphi, \vec{x}) = y$. Then define $y <_{\alpha+1} z$ iff $N(y) \tilde{<} N(z)$. Finally for limit λ set $<_\lambda = \bigcup_{\alpha<\lambda} <_\alpha$. Thus we obtain a wellordering $<_L = \bigcup_{\alpha \in \mathrm{ORD}} <_\alpha$ of L and a wellordering $\tilde{<}$ of names $(\alpha, \varphi, \vec{x})$ used to denote elements of L.

By an α–location we understand a location s of the form $s = (\alpha, \varphi, \vec{x})$. The $\tilde{<}$–smallest α–location is $(\alpha, \varphi_0, \vec{0})$ with $\vec{0}$ a vector of 0's of appropriate length. The $\tilde{<}$–successor of s is denoted by s^+.

2.1 Constructible Operations and Basic Constructible Closures.

The basic constructible operations are I and N as defined above and a Skolem function:

Interpretation. For a name $(\alpha, \varphi, \vec{x})$, set $I(\alpha, \varphi, \vec{x}) = \{y \in L_\alpha \mid L_\alpha \models \varphi(y, \vec{x})\}$.

Naming. For $y \in L$, let $N(y)$ be the $\tilde{<}$–least name $(\alpha, \varphi, \vec{x})$ such that $I(\alpha, \varphi, \vec{x}) = y$.

Skolem Function. For a name $(\alpha, \varphi, \vec{x})$, let $S(\alpha, \varphi, \vec{x})$ be the $<_L$– least $y \in L_\alpha$ such that $L_\alpha \models \varphi(y, \vec{x})$, and set $S(\alpha, \varphi, \vec{x}) = 0$ if such a y does not exist.

As we do not assume that α is a limit ordinal and therefore do not have pairing, we make the following nonstandard definition.

Definition: *For $X \subseteq L$ and $\vec{x}$ a finite sequence we write $\vec{x} \in X$ if each component of $\vec{x}$ belongs to X. If $(\alpha, \varphi, \vec{x})$ is a name we write $(\alpha, \varphi, \vec{x}) \in X$ to mean that $\alpha \in X$ and $\vec{x} \in X$.*
A set or class $X \subseteq L$ is constructibly closed, *written $X \triangleleft L$, iff X is closed under I, N and S, i.e.,*

$$(\alpha, \varphi, \vec{x}) \in X \longrightarrow I(\alpha, \varphi, \vec{x}) \in X \text{ and } S(\alpha, \varphi, \vec{x}) \in X,$$
$$y \in X \longrightarrow N(y) \in X.$$

For $X \subseteq L$ let $L\{X\}$ denote the $\subseteq$–smallest $Y \supseteq X$ such that $Y \triangleleft L$.

Clearly each L_α is constructibly closed.

Proposition 1 *Let X be constructibly closed and let $\pi: X \cong M$ be the Mostowski collapse of X onto the transitive set M. Then there is an ordinal α such that $M = L_\alpha$, and π preserves I, N, S and $<_L$:*

$$\pi: (X, \in, <_L, I, N, S) \cong (L_\alpha, \in, <_L, I, N, S).$$

Proof: We prove this for $X \subseteq L_\gamma$, by induction on γ. The cases $\gamma = 0$ and γ limit are easy. Let $\gamma = \beta + 1$ and $X \subseteq L_{\beta+1}$ but $X \not\subseteq L_\beta$. Closure under N and I implies that $X = \{I(\beta, \varphi, \vec{x}) \mid \vec{x} \text{ from } X \cap L_\beta\}$. Inductively let $\pi: X \cap L_\beta \cong L_\alpha$. Closure under S and the fact that β belongs to X imply that $X \cap L_\beta$ is elementary in L_β. It follows that π extends to $\tilde{\pi}: X \cong L_{\alpha+1}$. Preservation of I, N, S and $<_L$ follows also from the elementarity of $X \cap L_\beta$ in L_β. $\Box$

2.2 The Fine Hierarchy.

Definition: *Let s be a location, $s = (\alpha, \varphi_m, \vec{x})$. Set*

$$L_s = (L_\alpha, \in, <_L, I, N, S, S^{L_\alpha}_{\varphi_0}, S^{L_\alpha}_{\varphi_1}, \ldots, S^{L_\alpha}_{\varphi_m} \restriction \vec{x}, \emptyset, \emptyset, \ldots)$$

where $S^{L_\alpha}_{\varphi}(\vec{y}) = S(\alpha, \varphi, \vec{y})$, $S^{L_\alpha}_{\varphi_m} \restriction \vec{x}$ is the restricted Skolem function $S^{L_\alpha}_{\varphi_m} \restriction \{\vec{y} \mid \vec{y} <^{\text{lex}}_\alpha \vec{x}\}$ and $\emptyset, \emptyset, \ldots$ are empty functions.
$(L_s \mid s$ is a location$)$ is the fine constructible hierarchy.

Each structure of the fine hierarchy possesses an associated hull operator.

Definition: *Let $s = (\alpha, \varphi_m, \vec{x})$ be a location. A set $Y \subseteq L_\alpha$ is* closed *in L_s, written $Y \lhd L_s$, if Y is an algebraic substructure of L_s, i.e., if Y is closed under I, N, S, $S^{L_\alpha}_{\varphi_0}$, $S^{L_\alpha}_{\varphi_1}, \ldots, S^{L_\alpha}_{\varphi_m} \restriction \vec{x}$.*
*For a set $X \subseteq L_\alpha$ let $L_s\{X\}$ be the $\subseteq$–smallest set Y such that $Y \lhd L_s$ and $Y \supseteq X$. We call $L_s\{X\}$ the L_s–*hull *of X.*

The fine hierarchy is a very slow growing hierarchy which nonetheless satisfies full condensation. This is the basis for its applications to fine structure theory.

Proposition 2 (Condensation) *Let $s = (\alpha, \varphi_m, \vec{x})$ be a location and suppose X is a set such that $X \lhd L_s$.*
Then there is a unique isomorphism

$$\pi : (X, \in, <_L, I, N, S, S^{L_\alpha}_{\varphi_0}, S^{L_\alpha}_{\varphi_1}, \ldots, S^{L_\alpha}_{\varphi_m} \restriction \vec{x}, \emptyset, \ldots)$$
$$\cong L_{\overline{s}} = (L_{\overline{\alpha}}, \in, <_L, I, N, S, S^{L_{\overline{\alpha}}}_{\varphi_0}, S^{L_{\overline{\alpha}}}_{\varphi_1}, \ldots, S^{L_{\overline{\alpha}}}_{\varphi_{\overline{m}}} \restriction \vec{\overline{x}}, \emptyset, \ldots).$$

Proof: Let $\pi: X \cong L_{\overline{\alpha}}$ be given by Proposition 1. Note that X is φ_i–elementary in L_α for $i \leq m$, since X is closed under the Skolem functions for every proper subformula of φ_i. Hence $\pi^{-1}: L_{\overline{\alpha}} \to L_\alpha$ is φ_i–elementary for $i \leq m$. Let $r = (\overline{\alpha}, \varphi_i, \vec{w})$ be a location such that $\pi^{-1}(r) := (\alpha, \varphi_i, \pi^{-1}(\vec{w})) \tilde{<} (\alpha, \varphi_m, \vec{x})$. Then $z := S_{\varphi_i}^{L_\alpha}(\pi^{-1}(\vec{w}))$ belongs to X and $L_\alpha \models \varphi_i(z, \pi^{-1}(\vec{w}))$ iff $L_{\overline{\alpha}} \models \varphi_i(\pi(z), \vec{w})$. Moreover, if there is $\overline{z} \in L_{\overline{\alpha}}$ such that $L_{\overline{\alpha}} \models \varphi_i(\overline{z}, \vec{w})$, then $\pi(z)$ is the $<_L$–minimal such element, because $\overline{z} <_L \pi(z)$ and $L_{\overline{\alpha}} \models \varphi_i(\overline{z}, \vec{w})$ imply $L_\alpha \models \varphi_i(\pi^{-1}(\overline{z}), \pi^{-1}(\vec{w}))$ and $\pi^{-1}(\overline{z}) <_L z$, contradicting the definition of S_{φ_i}. Hence

$$\pi(z) = \pi(S_{\varphi_i}^{L_\alpha}(\pi^{-1}(\vec{w}))) = S_{\varphi_i}^{L_{\overline{\alpha}}}(\vec{w})$$

as required. The location $\overline{s}$ of the condensed structure is defined as the $\tilde{<}$–smallest strict upper bound of all r such that $\pi^{-1}(r) \tilde{<} s$ and $\overline{s} = \tilde{<}\text{–}\sup\{r \mid \pi^{-1}(r) \tilde{<} s\}$. □

Usually, we shall have $\overline{m} = m$ in the proposition, except when for every $\vec{w} \in L_{\overline{\alpha}}$ of the right length

$$\pi^{-1}(\vec{w}) <^{\text{lex}} \vec{x}.$$

In that case we have $\overline{m} = m + 1$ and $\vec{\overline{x}} = \vec{0}$, i.e., $\overline{s} = (\overline{\alpha}, \varphi_{m+1}, \vec{0})$ and

$$L_{\overline{s}} = (L_{\overline{\alpha}}, \in, <_L, I, N, S, S_{\varphi_0}^{L_{\overline{\alpha}}}, S_{\varphi_1}^{L_{\overline{\alpha}}}, \ldots, S_{\varphi_m}^{L_{\overline{\alpha}}}, \emptyset, \ldots)$$

observing that $S_{\varphi_{m+1}}^{L_{\overline{\alpha}}} \upharpoonright \vec{0} = \emptyset$.

The condensation situation in proposition 2 is often written as $\pi: X \cong L_{\overline{s}}$.

The slow growth of the $L_{\overline{s}}$–hierarchy is expressed by a finiteness property which says that at successor locations essentially only one more point enters the hulling process, and by continuity properties saying that at limit locations we just collect results of previous processes.

Proposition 3 (Finiteness Property) *Let $s = (\alpha, \varphi, \vec{x})$ be an α–location. Then there exists $z \in L_\alpha$ such that for any $X \subseteq L_\alpha$:*

$$L_{s^+}\{X\} \subseteq L_s\{X \cup \{z\}\}.$$

Proof: The expansion from L_s to L_{s^+} provides us with at most one new Skolem value in forming hulls, namely $S_\varphi^{L_\alpha}(\vec{x})$. Take this $S_\varphi^{L_\alpha}(\vec{x})$ to be z. □

Proposition 4 (Monotonicity) *(a) Suppose that* s_0 *and* s_1 *are* α*–locations such that* $s_0 \widetilde{\leq} s_1$. *Then* $L_{s_0}\{X\} \subseteq L_{s_1}\{X\}$ *for all* $X \subseteq L_{\alpha}$,.

(b) Suppose that α_0 *and* α_1 *are ordinals such that* $\alpha_0 < \alpha_1$. *If* s_0, s_1 *are* α_0*– and* α_1*–locations, respectively, and* $X \subseteq L_{\alpha_0}$ *then* $L_{s_0}\{X\} \subseteq L_{s_1}\{X \cup \{\alpha_0\}\}$.

Proof: Clear from the definitions. □

For the continuity property we have to distinguish between three kinds of limit locations:

Proposition 5 (Continuity)

(a) If α *is a limit ordinal,* $s = (\alpha, \varphi_0, \vec{0})$, *and* $X \subseteq L_\alpha$ *then*

$$L_s\{X\} = L\{X\} = \bigcup_{\beta<\alpha} L_{(\beta,\varphi_0,\vec{0})}\{X \cap L_\beta\}.$$

(b) If $s = (\alpha + 1, \varphi_0, \vec{0})$ *and* $X \subseteq L_\alpha$ *then*

$$L_s\{X \cup \{\alpha\}\} \cap L_\alpha = L\{X \cup \{\alpha\}\} \cap L_\alpha = \bigcup\{L_r\{X\} \mid r \textit{ is an } \alpha\textit{–location}\}.$$

(c) If $s = (\alpha, \varphi, \vec{x})$ *is a* $\widetilde{<}$*–limit,* $s \neq (\alpha, \varphi_0, \vec{0})$, *and* $X \subseteq L_\alpha$ *then*

$$L_s\{X\} = \bigcup\{L_r\{X\} \mid r \textit{ is an } \alpha\textit{–location, } r \widetilde{<} s\}.$$

Proof: (a) is clear from the definitions since the hull operators considered only use the functions I, N, S.

(b) The first equality is clear. The other is proved by two inclusions.

($\supseteq$) If z is an element of the right hand side, z is obtained from elements of X by successive applications of I, N, S and $S^{L_\alpha}_{\varphi_n}$ for $n < \omega$. Since $S^{L_\alpha}_{\varphi_n}(\vec{y}) = S(\alpha, \varphi_n, \vec{y})$, z is also obtainable from elements of $X \cup \{\alpha\}$ using only the I, N and S operations. Hence z belongs to the left hand side.

($\subseteq$) Conversely, consider $z \in L\{X \cup \{\alpha\}\} \cap L_\alpha$. There is a finite sequence computing z in $L\{X \cup \{\alpha\}\}$:

$$y_0, y_1, \ldots, y_k = z$$

such that each y_j is an element of $X \cup \{\alpha\}$ or y_j is obtained from $\{y_i \mid i < j\}$ by using I, N, S:

$$y_j = I(\beta, \varphi_n, \vec{y}) \quad \text{or} \quad y_j = S(\beta, \varphi_n, \vec{y}) \quad \text{or} \quad y_j \text{ is a component of } N(y)$$

for some $\beta, \vec{y}, y \in \{y_i \mid i < j\}$.
We show by induction on $j \leq k$:

$$\text{if } y_j \in L_\alpha \text{ then } y_j \in U = \bigcup\{L_r\{X\} \mid r \text{ is an } \alpha\text{–location}\}.$$

Case 1: $y_j \in X \cup \{\alpha\}$. Then our claim is obvious.
Case 2: $y_j = I(\beta, \varphi_n, \vec{y})$ (as in the first of the three ways of obtaining y_j from $\vec{y} \in \{y_i \mid i < j\}$, displayed above). If $\beta < \alpha$, then $\beta, \vec{y} \in U$ by the induction hypothesis and hence $y_j \in U$. If $\beta = \alpha$, then $\vec{y} \in U$ by the induction hypothesis. Setting

$$\psi(v, \vec{w}) = \forall u\, (u \in v \longleftrightarrow \varphi_n(u, \vec{w}))$$

with distinguished variable v we obtain $y_j = S_\psi^{L_\alpha}(\vec{y}) \in U$.
Case 3: $y_j = S(\beta, \varphi_n, \vec{y})$ (the second way of obtaining y_j). If $\beta < \alpha$, then $\beta, \vec{y} \in U$ and $y_j \in U$. If $\beta = \alpha$, then $\vec{y} \in U$ and $y_j = S_{\varphi_n}^{L_\alpha}(\vec{y}) \in U$.
Case 4: y_j is a component of $N(y_i)$ for some $i < j$ (the third way of obtaining y_j).
Case 4.1: $y_i \in L_\alpha$. Then $y_i \in U$ by the induction hypothesis. As U is closed under N, we get $N(y_i) \in U$, i.e., each component of $N(y_i)$ belongs to U.
Case 4.2: $y_i \in L_{\alpha+1} \setminus L_\alpha$. Then $y_i = \alpha$, or $y_i = I(\alpha, \psi, \vec{y})$ for some $\vec{y} \in \{y_h \mid h < i\}$. Since $\alpha = I(\alpha, \text{“}u \text{ is an ordinal”}, \emptyset)$, we may assume the latter. $N(y_i)$ will be of the form $(\alpha, \chi, (c_0, \ldots, c_{m-1}))$. We obtain c_0 in U as follows: If

$$\chi_0(v_0, \vec{w}) = \exists v_1 \ldots \exists v_{m-1} \forall u\, (\chi(u, v_0, v_1, \ldots, v_{m-1}) \longleftrightarrow \psi(u, \vec{w}))$$

with distinguished variable v_0 then $c_0 = S_{\chi_0}^{L_\alpha}(\vec{y}) \in U$, since, inductively, $\vec{y} \in U$. We obtain c_1 in U as follows: If

$$\chi_1(v_1, \vec{w}) = \exists v_2 \ldots \exists v_{m-1} \forall u\, (\chi(u, v_0, v_1, \ldots, v_{m-1}) \longleftrightarrow \psi(u, \vec{w}))$$

with distinguished variable v_1 then $c_1 = S_{\chi_1}^{L_\alpha}(c_0{}^\frown\vec{y}) \in U$. Proceeding like this we see that $y_j \in U$.
(c) is again obvious from the definitions. □

This completes our list of basic properties of the hull operations associated with the fine hierarchy. They are sufficient to establish Jensen's Square Principle in L, which we consider next.

3 A Proof of Square

Theorem (Jensen): *Assume $V = L$. There exists a sequence $\langle C_\beta \mid \beta$ singular $\rangle$ such that*
(a) C_β is closed unbounded in β,
(b) C_β has ordertype less than β,
(c) if $\overline{\beta}$ is a limit point of C_β then $\overline{\beta}$ is singular and $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$.

Proof: Let β be singular. The following claim gives a reformulation of the singularity of β:

Claim 1: *There is a location $s = (\gamma, \varphi, \vec{x})$, $\gamma \geq \beta$, and a finite set $p \subseteq L_\gamma$ such that*

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}$$

is bounded in β.

Proof: Choose α less than β and a function $f: \alpha \to \beta$ cofinally. Choose $\gamma \in \mathrm{ORD}$ such that $f \in L_\gamma$. Set $p = \{f\}$ and $s = (\gamma, \varphi_{n+1}, \vec{0})$ where n is a natural number choosen such that $\varphi_n \equiv v_0 = v_1(v_2)$ with distinguished variable v_0. If $\alpha \leq \overline{\beta} < \beta$ then

$$\beta \cap L_s\{\overline{\beta} \cup p\} \supseteq \beta \cap L_s\{\alpha \cup p\} \supseteq f''\alpha.$$

Hence $\beta \cap L_s\{\overline{\beta} \cup p\}$ is cofinal in β, and so $\beta \cap L_s\{\overline{\beta} \cup p\} \neq \overline{\beta}$. $\Box$ (Claim 1)

Let $s = s(\beta)$ be $\tilde{<}$–minimal satisfying Claim 1, together with the finite set $p \subseteq L_\gamma$. We show that s is a $\tilde{<}$–limit which can be nicely approximated from below.

Claim 2: *s is a limit location.*

Proof: Assume that $s = r^+$. By the Finiteness Property (Proposition 3) there exists a $z \in L_\gamma$ such that if $\overline{\beta}$ is less than β then

$$L_s\{\overline{\beta} \cup p\} \subseteq L_r\{\overline{\beta} \cup p \cup \{z\}\}.$$

So

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_r\{\overline{\beta} \cup p \cup \{z\}\}\} \subseteq \{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}.$$

Hence $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_r\{\overline{\beta} \cup p \cup \{z\}\}\}$ is bounded in β, contradicting the minimality of s. $\Box$ (Claim 2)

Claim 3: $s \neq (\beta, \varphi_0, \vec{0})$.

Proof: Assume that $s = (\beta, \varphi_0, \vec{0})$. Choose β_0 less than β such that $p \subseteq L_{\beta_0}$. If $\beta_0 \leq \overline{\beta} < \beta$ then

$$\overline{\beta} \subseteq \beta \cap L_s\{\overline{\beta} \cup p\} \subseteq \beta \cap L\{\overline{\beta} \cup p\} \subseteq \beta \cap L_{\overline{\beta}} = \overline{\beta},$$

contradicting the fact that s and p satisfy the requirements in Claim 1. □ (Claim 3)

Claim 4: $s \neq (\gamma, \varphi_0, \vec{0})$ *for limit* γ.

Proof: Assume that there is a limit ordinal γ such that $s = (\gamma, \varphi_0, \vec{0})$. Choose γ_0 less than γ such that $p \subseteq L_{\gamma_0}$ and $\gamma_0 \geq \gamma$, and set $s_0 = (\gamma_0, \varphi_0, \vec{0})$. Then

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_{s_0}\{\overline{\beta} \cup p\}\} \subseteq \{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}.$$

Hence $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_{s_0}\{\overline{\beta} \cup p\}\}$ is bounded below β, contradicting the minimality of s. □ (Claim 4)

In defining C_β we shall consider three special cases and a generic case. In the special cases, β will have cofinality ω and we can pick any ω–sequence cofinal in β as C_β.

Special Case 1. $s = (\alpha + 1, \varphi_0, \vec{0})$ *for some* α.

Every element of $L_{\alpha+1}$ can be "named" by α and finitely many elements of L_α. So we may assume that p is of the form $p = q \cup \{\alpha\}$ with $q \subseteq L_\alpha$.
Define a strictly increasing sequence $(\beta_n \mid n < \omega)$ of ordinals less than β recursively: Let

$$\beta_0 = \max\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\} < \beta.$$

Given β_n choose β_{n+1} greater than β_n least such that

$$\beta_{n+1} = \beta \cap L_{(\alpha, \varphi_n, \vec{0})}\{\beta_{n+1} \cup q\}.$$

Since $s = (\alpha, \varphi_n, \vec{0}) \mathrel{\tilde{<}} (\alpha+1, \varphi_0, \vec{0})$, the definition of s implies that β_{n+1} exists below β. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$. Then

$$\begin{aligned}\beta \cap L_s\{\beta_\omega \cup p\} &= \beta \cap L_s\{\beta_\omega \cup q \cup \{\alpha\}\} \\ &= \beta \cap \bigcup\{L_r\{\beta_\omega \cup q\} \mid r \text{ is an } \alpha\text{–location}\} \\ &= \bigcup_{n<\omega} \beta \cap L_{(\alpha,\varphi_n,\vec{0})}\{\beta_\omega \cup q\} \\ &= \bigcup_{n<\omega} \beta \cap L_{(\alpha,\varphi_n,\vec{0})}\{\beta_{n+1} \cup q\} \\ &= \bigcup_{n<\omega} \beta_{n+1} = \beta_\omega;\end{aligned}$$

the second equality uses Proposition 5(b), the third and fourth use the monotonicity property of our hulls (Proposition 4(a)). Now by the definition of β_0 we must have $\beta_\omega = \beta$. Hence setting

$$C_\beta = \{\beta_n \mid n < \omega\}$$

we get a cofinal subset of β. This finishes Special Case 1.

Now assume that $s = (\gamma, \varphi, \vec{x}) \neq (\gamma, \varphi_0, \vec{0})$.

Claim 5: *There is a finite $\overline{p} \subseteq L_\gamma$ such that $L_s\{\beta \cup \overline{p}\} = L_\gamma$.*

Proof: By condensation (Proposition 2), there are a unique function π and a unique location $\overline{s}$ such that $\pi\colon L_s\{\beta \cup p\} \cong L_{\overline{s}}$. Then we have $L_{\overline{s}} = L_{\overline{s}}\{\beta \cup \overline{p}\}$ where $\overline{p} = \pi''p$. As $\pi \upharpoonright \beta = \mathrm{id}$, we can conclude that $\beta \cap L_s\{\overline{\beta} \cup p\} = \beta \cap L_{\overline{s}}\{\overline{\beta} \cup \overline{p}\}$ holds for all $\overline{\beta}$ less than β. Hence

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_{\overline{s}}\{\overline{\beta} \cup \overline{p}\}\} = \{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}$$

is bounded below β. Then $\overline{s} = s$ by the $\tilde{<}$–minimality of s, and so $L_s = L_s\{\beta \cup \overline{p}\} = L_\gamma$. □ (Claim 5)

Let $<^*$ be the canonical wellorder of finite subsets of L derived from $<_L$: $p_0 <^* p_1 \longleftrightarrow p_0 \neq p_1$ and the $<_L$-maximal element of $p_0 \triangle p_1$ belongs to p_1. Choose a $<^*$-minimal $p(\beta) \subseteq L_\gamma$ such that $p(\beta)$ satisfies Claim 5. Since in particular the old parameter p is generated by $\beta \cup p(\beta)$ we have

Claim 6: $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p(\beta)\}\}$ *is bounded below β. Let $\beta_0 < \beta$ be the maximum of this set.*

By Claim 6, $p(\beta)$ satisfies the requirements in Claim 1 and we may denote $p(\beta)$ by p without danger of confusion.
We have to examine which locations below s are computed in $L_s\{X\}$: for $Y \subseteq L_\gamma$ we write $r = (\gamma, \psi, \vec{y}) \in Y$ if $\vec{y} \in Y$. We say that a subset Y of L_γ is *bounded below* s, if there is $s_0 \tilde{<} s$ such that if $r \tilde{<} s$ and $r \in Y$, then $r \tilde{<} s_0$. The $\tilde{<}$-least such s_0 is called the *$\tilde{<}$-least upper bound of Y below s.* Note that if in addition $Y = L_s\{Z\}$ then we get $L_s\{Z\} = L_{s_0}\{Z\}$.

Special Case 2. *$L_s\{\alpha \cup p\}$ is bounded below s for every $\alpha < \beta$.*

Define a strictly increasing sequence $(\beta_n \mid n < \omega)$ of ordinals less than β recursively: Let β_0 be defined as in Claim 6. Given β_n, set

$$\beta_{n+1} = \bigcup(\beta \cap L_s\{(\beta_n + 1) \cup p\}).$$

By Special Case 2, there is $r \tilde{<} s$ such that

$$L_s\{(\beta_n + 1) \cup p\} = L_r\{(\beta_n + 1) \cup p\}.$$

The minimality of s implies that $\beta \cap L_r\{(\beta_n + 1) \cup p\}$ cannot be cofinal in β, and so β_{n+1} is less than β. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$. Then

$$\beta_\omega \subseteq \beta \cap L_s\{\beta_\omega \cup p\} \subseteq \bigcup_{n<\omega} \beta \cap L_s\{(\beta_n + 1) \cup p\} \subseteq \bigcup_{n<\omega} \beta_{n+1} = \beta_\omega,$$

and since β_ω is greater than β_0 we have $\beta_\omega = \beta$. Hence setting

$$C_\beta = \{\beta_n \mid n < \omega\}$$

we get a cofinal subset of β. This finishes Special Case 2.

Now assume that $L_s\{\alpha_0 \cup p\}$ is unbounded below s for some α_0 less than β. Choose $\alpha_0 = \alpha_0(\beta)$ least with this property. We would like to use α_0 to steer the singularisation of β and obtain ordertype$(C_\beta) \leq \max\{\alpha_0, \omega\} < \beta$. If α_0 is neither a limit ordinal nor zero we have to look for another steering ordinal. In this case we write $\alpha_0 = \alpha_0' + 1$, and we choose a least $\alpha_1 = \alpha_1(\beta)$ less than α_0 such that

$$L_s\{\alpha_1 \cup p \cup \{\alpha_0'\}\}$$

is unbounded below s. If $\alpha_1 = \alpha_1' + 1$, then we choose a least $\alpha_2 = \alpha_2(\beta)$ less than α_1 such that

$$L_s\{\alpha_2 \cup p \cup \{\alpha_0', \alpha_1'\}\}$$

is unbounded below s. Continuing this way we find a natural number $k = k(\beta)$ such that $\alpha = \alpha(\beta) = \alpha_k(\beta)$ is a limit ordinal or zero.

Special Case 3. $\alpha = 0$.

One easily sees that $L_s\{p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is a countable set. Since $\alpha = 0$, it is unbounded below s. So s has "cofinality ω" in the ordering of locations and we can find a strictly increasing sequence $(s_n \mid n < \omega)$ of γ–locations converging towards s. Define a strictly increasing sequence $(\beta_n \mid n < \omega)$ of ordinals less than β recursively: Let β_0 be defined as in Claim 6. Given β_n, choose β_{n+1} greater than β_n minimal such that

$$\beta_{n+1} = \beta \cap L_{s_{n+1}}\{\beta_{n+1} \cup p\}.$$

β_{n+1} exists, since $s_{n+1} \tilde{<} s$. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$. Then

$$\beta_\omega = \bigcup_{n<\omega} \beta_{n+1} = \bigcup_{n<\omega} \beta \cap L_{s_{n+1}}\{\beta_{n+1} \cup p\} = \beta \cap L_s\{\beta_\omega \cup p\},$$

hence the definition of β_0 implies $\beta_\omega = \beta$. Setting

$$C_\beta = \{\beta_n \mid n < \omega\}$$

we get a cofinal subset of β. This finishes Special Case 3.

So, finally, we arrive at the generic case:

Generic Case. $s = (\gamma, \varphi, \vec{x}) \neq (\gamma, \varphi_0, \vec{0})$, *and* $L_s\{\alpha \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ *is unbounded below* s *where* α *is a limit ordinal less than* β.

Define sequences $(\beta_i(\beta) \mid i \leq \alpha)$ and $(s_i \mid 0 < i \leq \alpha)$ recursively: Let $\beta_0 < \beta$ be defined as in Claim 6. For each $0 < i \leq \alpha$ let s_i be the $\tilde{<}$–least upper bound of $L_s\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below s, and let $\beta_i = \beta_i(\beta)$ be the least ordinal greater than β_0 such that

$$\beta_i = \beta \cap L_{s_i}\{\beta_i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}.$$

If $i < \alpha$ then $\beta_i < \beta$ because $s_i \tilde{<} s$; also $s_\alpha = s$, $\beta_\alpha = \beta$ and

Claim 7: *If* $0 < i < j < \alpha$ *then* $s_i \tilde{\leq} s_j$ *and* $\beta_i \leq \beta_j$.

Claim 8: $\{\beta_i \mid i < \alpha\}$ *is closed unbounded in* β.

Proof: Let $\overline{\alpha} \leq \alpha$ be a limit ordinal. We only have to show that $\beta_{\overline{\alpha}} = \bigcup_{i<\overline{\alpha}} \beta_i$ and since $\beta_{\overline{\alpha}} \geq \beta_i$ for $i < \overline{\alpha}$ it suffices to see that

$$\bigcup_{i<\overline{\alpha}} \beta_i = \bigcup_{i<\overline{\alpha}} \beta \cap L_{s_i}\{\beta_i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$$
$$= \beta \cap L_{s_{\overline{\alpha}}}\{\textstyle\bigcup_{i<\overline{\alpha}} \beta_i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$$

so that $\bigcup_{i<\overline{\alpha}} \beta_i$ satisfies the defining property of $\beta_{\overline{\alpha}}$. □ (Claim 8)

C_β will now be defined as an endsegment of such β_i's for which important elements of the preceding construction are visible below β_i or s_i. Let $I(\beta)$ be the set of those ordinals i that satisfy the following properties (1) — (5):

(1) $0 < i < \alpha$, and if $l \leq k$ then $\beta_i \geq \alpha'_l$.

(2) s_i is a γ–location.

(3) $j < \beta_j$ for $i \leq j < \alpha$.

(4) If $l < k$ and t is the $\tilde{<}$–least upper bound of $L_s\{\alpha'_l \cup p \cup \{\alpha'_0, \ldots, \alpha'_{l-1}\}\}$ below s then $s_i \tilde{>} t$.

(5) If $\beta < \gamma$ then $\beta \in L_{s_i}\{\beta_i \cup p\}$.

Using the following facts (i) — (iv) the reader can easily show that there is i_0 less than α such that an ordinal i less than α satisfies the conditions (1) — (5) if and only if $i > i_0$, i.e., $I(\beta)$ is a final segment of α.

(i) $L_s\{\alpha \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below s.

(ii) $\alpha < \beta$ and $\beta = \bigcup\{\beta_i | i < \alpha\}$ where $(\beta_i \mid i < \alpha)$ is (weakly) increasing.

(iii) $L_s\{\alpha'_l \cup p \cup \{\alpha'_0, \ldots, \alpha'_{l-1}\}\}$ is bounded below s for all $l \leq k$..

(iv) If $\beta < \gamma$ then $\beta \in L_s\{\beta \cup p\} = L_\gamma$.

So set

$$C_\beta = \{\beta_i \mid i \in I(\beta)\}.$$

Then

Claim 9: *C_β is closed unbounded in β and* ordertype$(C_\beta) \leq \alpha < \beta$.

This completes the definition of the system $\langle C_\beta \mid \beta \text{ singular}\rangle$, and we are left with proving the coherence property. Fix $\overline{\beta}$ less than β such that $\overline{\beta}$ is a limit point of C_β. We have to show that $\overline{\beta}$ is singular and $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$. β falls under the Generic Case, as ordertype$(C_\beta) > \omega$. Let $\overline{\alpha}$ be the least ordinal η such that $\overline{\beta} = \beta_\eta$. Then $\overline{\alpha}$ is a limit ordinal and $\overline{\beta}$ is singular since $\mathrm{cf}(\beta_{\overline{\alpha}}) \leq \overline{\alpha} < \beta_{\overline{\alpha}}$. By condensation there is an isomorphism

$$\pi\colon L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\} \cong L_{\overline{s}}.$$

Let $q = \pi'' p$ and $\overline{\gamma} = \alpha(\overline{s})$.

Claim 10: $\pi \restriction \overline{\beta} = \mathrm{id}$. *If s is a β–location then $\overline{s}$ is a $\overline{\beta}$–location while if s is a γ–location and $\gamma > \beta$ then $\pi(\beta) = \overline{\beta}$.*

Proof: If $\gamma > \beta$ then $\beta \in L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\}$ and $\overline{\beta} = \beta \cap L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\}$. $\square$ (Claim 10)

Claim 11: $\overline{s} = s(\overline{\beta})$.

Proof: If $\beta_0 < \delta < \overline{\beta}$ then $\delta \neq \beta \cap L_{s_{\overline{\alpha}}}\{\delta \cup p \cup \{\alpha'_0 \dots \alpha'_{k-1}\}\}$ and therefore $\delta \neq \overline{\beta} \cap L_{\overline{s}}\{\delta \cup q \cup \{\alpha'_0 \dots \alpha'_{k-1}\}\}$. It follows that $s(\overline{\beta}) \mathrel{\tilde{\leq}} \overline{s}$.
Conversely if $r \mathrel{\tilde{<}} \overline{s}$ and $\overline{q}$ is a finite subset of $L_{\alpha(r)}$ then $\pi^{-1}(r) \mathrel{\tilde{<}} s_i$ and $\pi^{-1\prime\prime}\overline{q} \subseteq L_{s_i}\{\beta_i \cup p\}$ for sufficiently large i less than $\overline{\alpha}$, since the s_i's are unbounded below $s_{\overline{\alpha}}$, the β_i's are unbounded in $\overline{\beta}$ and $L_{\overline{s}}\{\overline{\beta} \cup q\} = L_{\alpha(\overline{s})}$. As $\beta_i = \beta \cap L_{s_i}\{\beta_i \cup p\}$ we get $\beta_i = \overline{\beta} \cap L_r\{\beta_i \cup \overline{q}\}$ for β_i's cofinal in $\overline{\beta}$ and so $r \mathrel{\tilde{<}} s(\overline{\beta})$. Therefore $\overline{s} \mathrel{\tilde{\leq}} s(\overline{\beta})$. $\square$ (Claim 11)

Claim 12: *$\overline{\beta}$ does not fall under Special Case 1.*

Claim 13: $q = p(\overline{\beta})$.

Proof: As $L_{\overline{s}}\{\overline{\beta} \cup q\} = L_{\overline{\gamma}}$, we get $q \geq^* p(\overline{\beta})$. Assume $q >^* p(\overline{\beta})$. As $p(\overline{\beta})$ satisfies the requirements in Claim 5 at $\overline{\beta}$, we get $q \subseteq L_{\overline{s}}\{\overline{\beta} \cup p(\overline{\beta})\}$, hence $p = \pi^{-1\prime\prime} q \subseteq L_s\{\overline{\beta} \cup \pi^{-1\prime\prime} p(\overline{\beta})\}$. So $\pi^{-1\prime\prime} p(\overline{\beta}) <^* p = \pi^{-1\prime\prime} q$ and $\pi^{-1\prime\prime} p(\overline{\beta})$ satisfies the requirements in Claim 5, contrary to the minimal choice of $p = p(\beta)$. $\square$ (Claim 13)

Now $L_{s_{\overline{\alpha}}}\{\overline{\alpha} \cup p\} = L_s\{\overline{\alpha} \cup p\}$ is unbounded below $s_{\overline{\alpha}}$. Hence $L_{\overline{s}}\{\overline{\alpha} \cup q\}$ is unbounded below $\overline{s}$, and $\overline{\alpha} < \overline{\beta}$. Hence

Claim 14: *$\overline{\beta}$ does not fall under Special Case 2.*

Claim 15: *If $j < k$ then $\alpha_j(\beta) = \alpha_j(\overline{\beta})$.*

Proof: By induction on $j < k$.
By definition, $\alpha_j(\beta)$ is the smallest ν s.t. $L_s\{\nu \cup p \cup \{\alpha'_i \mid i < j\}\}$ is unbounded below s. Now $L_s\{\overline{\alpha} \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below $s_{\overline{\alpha}}$, so $L_{\overline{s}}\{\overline{\alpha} \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below $\overline{s}$. Hence $L_{\overline{s}}\{\alpha_j(\beta) \cup q \cup \{\alpha'_0, \ldots, \alpha'_{j-1}\}\}$ is unbounded below $\overline{s}$, as $\overline{\alpha} \cup \{\alpha'_j \ldots \alpha'_{k-1}\} \subseteq \alpha_j(\beta)$. Conversely, the definition of $I(\beta)$ implies that $L_s\{\alpha'_j \cup p \cup \{\alpha'_0, \ldots, \alpha'_{j-1}\}\}$ is bounded below s by some $s' \tilde{<} s_{\overline{\alpha}}$, hence by some location in $L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\}$. So $L_{\overline{s}}\{\alpha'_j \cup q \cup \{\alpha'_0, \ldots, \alpha'_{j-1}\}\}$ is bounded below $\overline{s}$ by some location less than $\overline{s}$. So $\alpha_j(\beta) = \alpha_j(\overline{\beta})$. □ (Claim 15)

Claim 16: $\alpha_k(\overline{\beta}) = \overline{\alpha}$.

Proof: The set $L_{\overline{s}}\{\overline{\alpha} \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below $\overline{s}$. If we take α' less than $\overline{\alpha}$, then $L_{s_{\overline{\alpha}}}\{\alpha' \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is bounded below $s_{\overline{\alpha}}$, by the minimality of $\overline{\alpha}$. So we have $\alpha_k(\overline{\beta}) = \overline{\alpha}$. □ (Claim 16)

Claim 17: *$\overline{\beta}$ does not fall under Special Case 3,*

since $\overline{\alpha} \neq 0$. So we are again in the Generic Case.

Claim 18: *If $i < \overline{\alpha}$ then $\beta_i(\beta) = \beta_i(\overline{\beta})$.*

Proof: By definition, $\beta_0 = \beta_0(\beta)$ is the largest δ less than β such that $\delta = \beta \cap L_s\{\delta \cup p\}$. From the definition of $\overline{\beta} = \beta_{\overline{\alpha}}$ we infer that β_0 is the largest δ less than $\overline{\beta}$ such that $\delta = \overline{\beta} \cap L_{s_{\overline{\alpha}}}\{\delta \cup p\}$. As $L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\} \cong L_{\overline{s}}\{\overline{\beta} \cup q\}$ by a map which is the identity on $\overline{\beta}$, we see that β_0 is the largest δ less than $\overline{\beta}$ such that $\delta = \overline{\beta} \cap L_{\overline{s}}\{\delta \cup q\}$, which is the definition of $\beta_0(\overline{\beta})$.
Now consider $0 < i < \overline{\alpha}$. Then

$s_i(\beta)$ is the $\tilde{<}$–least upper bound of $L_s\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below s.

By the definition of $s_{\overline{\alpha}}$ we get that

$s_i(\beta)$ is the $\tilde{<}$–least upper bound of $L_{s_{\overline{\alpha}}}\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below $s_{\overline{\alpha}}$.

Moreover,

$s_i(\overline{\beta})$ is the $\tilde{<}$–least upper bound of $L_{\overline{s}}\{i \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below $\overline{s}$.

Now $\beta_i(\beta)$ is the minimal ordinal greater than β_0 such that

$$\beta_i(\beta) = \beta \cap L_{s'}\{\beta_i(\beta) \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$$

for all $s' \tilde{<} s_{\overline{\alpha}}(\beta)$ with $s' \in L_{s_{\overline{\alpha}}}\{i \cup p \cup \{\alpha'_0 \ldots \alpha'_{k-1}\}\}$, and $\beta_i(\overline{\beta})$ is the minimal ordinal greater than β_0 such that

$$\beta_i(\overline{\beta}) = \overline{\beta} \cap L_{\overline{s}'}\{\beta_i(\overline{\beta}) \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$$

for all $\overline{s}' \tilde{<} \overline{s}$ with $\overline{s}' \in L_{\overline{s}}\{i \cup q \cup \{\alpha'_0 \ldots \alpha'_{k-1}\}\}$. By the above and the fact that $\pi \restriction \overline{\beta} = \mathrm{id}$ we have $\beta_i(\beta) = \beta_i(\overline{\beta})$ as required. $\square$ (Claim 18)

Now one easily checks that each ordinal i less than $\overline{\alpha}$ satisfies the defining properties of $I(\beta)$ (cf. (1) — (5) above) if and only if it satisfies the corresponding defining properties of $I(\overline{\beta})$. So we get $I(\overline{\beta}) = I(\beta) \cap \overline{\alpha}$, and this immediately implies the coherence property. $\square$

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