

Fine Structure Theory and Its Applications

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LECTURE 1. INTRODUCTION, THE DIAMOND PRINCIPLE

The fine structure of L is an exciting theory due almost entirely to the efforts of Ronald Jensen. By developing a thorough analysis of definability in L , he established combinatorial principles which can be used to solve many important problems in set theory, under the hypothesis $V = L$.

These lectures constitute an introduction to this work from a recursion-theorist's point of view. This is not wholly inappropriate, for a recursion-theoretic spirit is prevalent throughout the fine structure theory. The essential intuition is to view Σ_1 -definability as a generalized form of recursive enumerability, an idea which has also been key to the development of higher recursion theory.

Thus a recursion-theoretic idea has been of importance in a set-theoretic context. In recent years there has been a significant flow in the opposite direction as well. It is our purpose in these lectures to describe the main techniques of the fine structure theory and to describe how these techniques have been applied to recursion theory.

The starting point for any discussion of the fine structure of L must be the Gödel Collapse Lemma. The uniformity of the L -hierarchy which is illustrated by this lemma is of fundamental importance.

We assume basic familiarity with Gödel's L -hierarchy $\langle L_\alpha \mid \alpha \in ORD \rangle$. Now fix a limit ordinal λ and suppose that $a \in L_\lambda$, $X \subseteq L_\lambda$. We say that a is *definable in L_λ* with parameters from X if for some formula in set theory $\phi(v)$ with parameters from X , $L_\lambda \models$ " a is the unique solution to $\phi(v)$ ". We write $a \in H_\lambda(X)$ in this case.

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An alternative description of $H_\lambda(X)$ can be provided using Skolem functions. For each formula $\phi(x, x_1, \dots, x_n)$ let f_ϕ be defined on

$$\overbrace{L_\lambda \times \dots \times L_\lambda}^{n \text{ times}}$$

by $f_\phi(x_1, \dots, x_n) = <_{L_\lambda}$ -least $x \in L_\lambda$ such that $L_\lambda \models \phi(x, x_1, \dots, x_n)$ if such an x exists; $= 0$ otherwise. Then $H_\lambda(X) =$ closure of X under the f_ϕ 's. It follows that $\langle H_\lambda, \varepsilon \rangle < \langle L_\lambda, \varepsilon \rangle$.

GÖDEL COLLAPSE LEMMA. *For all $X \subseteq L_\lambda$, $\langle H_\lambda(X), \varepsilon \rangle$ is isomorphic to $\langle L_\beta, \varepsilon \rangle$ for some unique $\beta \leq \lambda$. If $t \subseteq H_\lambda(X)$ is transitive then the isomorphism is the identity on t .*

By the Mostowski Isomorphism Theorem we know that $\langle H_\lambda(X), \varepsilon \rangle$ is isomorphic to $\langle T, \varepsilon \rangle$ for some transitive T . Gödel absoluteness shows that there is a sentence ϕ of set theory such that for transitive T , $\langle T, \varepsilon \rangle \models \phi$ iff $T = L_\beta$ for some β . This proves the first statement of the Gödel Collapse Lemma. The second statement follows from the fact that any isomorphism of transitive sets is the identity.

The original use of the Gödel Collapse Lemma was to establish the Generalized Continuum Hypothesis (GCH) in L . It is worthwhile to review that argument as it is the model for much of what comes later.

THEOREM (GÖDEL). *Assume $V = L$ and suppose κ is an infinite cardinal, $\gamma < \kappa^+$. If $A \subseteq \gamma$ then $A \in L_{\kappa^+}$.*

PROOF OF THEOREM. Pick a limit λ so that $A \in L_\lambda$. Then by the Collapse Lemma, $\pi: H_\lambda(\gamma \cup \{A\}) \simeq L_\beta$ for some π, β and $\pi \upharpoonright \gamma = \text{id} \upharpoonright \gamma$. (We are dropping the ε -relation from our structures to save writing.) But then $\pi(A) = A$ so $A \in L_\beta$. As $\text{card}(H_\lambda(\gamma \cup \{A\})) \leq \kappa$ we have $\text{card}(L_\beta) = \text{card}(\beta) \leq \kappa$ so $\beta < \kappa^+$. So $A \in L_{\kappa^+}$. Q.E.D.

Thus we see that the Gödel Collapse Lemma enforces a restriction on the possible subsets of an infinite cardinal κ , the GCH being a consequence of this restriction. By taking a closer look at this technique we can uncover a deeper type of restriction, which is embodied in Jensen's Diamond Principle. This type of restriction is most easily described by making use of the notion of "cutoff" function.

DEFINITION. Suppose $V = L$, κ an infinite cardinal and $A \subseteq \kappa^+$. Define $f_A: \kappa^+ \rightarrow \kappa^+$ (the cutoff function of A) by

$$f_A(\gamma) = L\text{-rank}(A \cap \gamma) = \text{least } \delta (A \cap \gamma \in L_\delta).$$

Note that by the previous Theorem, $f_A(\gamma) < \kappa^+$ for all $\gamma < \kappa^+$, $C \subseteq \kappa^+$ is *closed unbounded* (CUB) if $\sup C = \kappa^+$, and for all $\gamma < \kappa^+$, $\sup(C \cap \gamma) \in C$.

THEOREM (JENSEN). *Assume $V = L$. There is a fixed $f^*: \kappa^+ \rightarrow \kappa^+$ such that for all $A \subseteq \kappa^+$, f^* dominates f_A on a CUB set; i.e., $\{\gamma | f^*(\gamma) \geq f_A(\gamma)\}$ contains a CUB set for each $A \subseteq \kappa^+$.*

Note. Of course, the CUB set depends on A !
 PROOF. $f^*(\gamma) = \text{least } \gamma' (L_{\gamma'} \models \text{card}(\gamma) \leq \kappa)$. Now suppose that $A \subseteq \kappa^+$ and pick a limit λ , $A \in L_\lambda$. For any γ between κ and κ^+ we may form $H_\lambda(\gamma \cup \{A\}) = H(\gamma)$.

Claim. $H(\gamma) \cap \kappa^+ = \text{an ordinal } \gamma_A$.

PROOF OF CLAIM. We have to show that if $\delta \in H(\gamma) \cap \kappa^+$ then $\delta \subseteq H(\gamma)$. As $H(\gamma) < L_\lambda$ we have $g: \kappa \leftrightarrow \delta$, $g \in H(\gamma)$. As $\kappa \subseteq H(\gamma)$ it follows that $g[\kappa] = \delta \subseteq H(\gamma)$. Q.E.D. (of Claim)

The reason why the ordinals γ_A are useful is that $f^*(\gamma_A) > f_A(\gamma_A)$: Indeed we have $\pi: H(\gamma) \simeq L_\beta$ where $\pi \upharpoonright \delta_A = \text{id} \upharpoonright \gamma_A$. So $\pi(A) = \{\pi(\delta) \mid \delta < \gamma_A\} = A \cap \gamma_A \in L_\beta$. Thus $f_A(\gamma_A) \leq \beta$, by definition of f_A . But $\beta < f^*(\gamma_A)$ as $\gamma_A = \pi(\kappa^+) = (\kappa^+)^{L_\beta}$.

To complete the proof we need only check that $D = \{\gamma_A \mid \kappa < \gamma < \kappa^+\}$ contains a CUB set. In fact D is CUB: Clearly D is unbounded in κ^+ . Suppose δ is a limit of elements of D , $\delta < \kappa^+$. Then $\delta_A = \sup_{\gamma < \delta} \gamma_A = \sup_{\gamma_A < \delta} (\gamma_A)_A$. But as $H_\lambda(\gamma \cup \{A\}) = H_\lambda(\gamma_A \cup \{A\})$ for all γ , we have $(\gamma_A)_A = \gamma_A$. So $\delta_A = \sup_{\gamma_A < \delta} \gamma_A = \delta$ and $\delta \in D$. Q.E.D.

We can now convert what we have into a tidy combinatorial principle:

$\diamond_{\kappa^+}^*$: There exists $\langle D_\gamma \mid \gamma < \kappa^+ \rangle$ such that:

(a) For each $\lambda < \kappa^+$, $D_\lambda \subseteq \mathcal{P}(\lambda)$, $\text{card}(D_\lambda) \leq \kappa$.

(b) For any $A \subseteq \kappa^+$, $\{\gamma \mid A \cap \gamma \in D_\gamma\}$ contains a CUB set.

$\diamond_{\kappa^+}^*$ is obtained by letting $D_\gamma = \mathcal{P}(\gamma) \cap L_{f^*(\gamma)}$.

The \diamond_{κ^+} -principle is a variant of the above: Say that $X \subseteq \kappa^+$ is *stationary* if $X \cap C \neq \emptyset$ whenever $C \subseteq \kappa^+$ is CUB. Then \diamond_{κ^+} is obtained from $\diamond_{\kappa^+}^*$ by replacing “ $\text{card}(D_\gamma) \leq \kappa$ ” by “ $\text{card}(D_\gamma) = 1$ ” and “CUB” by “stationary”. A lemma of Kunen shows that $\diamond_{\kappa^+}^* \rightarrow \diamond_{\kappa^+}$ (in ZFC). (In fact, \diamond_{κ^+} is equivalent to \diamond_{κ^+}' , where only the second of the above two changes is made.) A stronger principle, $\diamond_{\kappa^+}^+$ also requires that the CUB set in (b) have the property $\gamma \in C \rightarrow C \cap \gamma \in D_\gamma$. Our proof of $\diamond_{\kappa^+}^*$, in fact, demonstrates $\diamond_{\kappa^+}^+$ as well. There are also versions of these principles for inaccessible cardinals. (In the case of \diamond_κ^* , \diamond_κ^+ one must require $\text{card}(D_\gamma) \leq \text{card}(\gamma)$ for $\gamma < \kappa$.) As it turns out, if $V = L$ then \diamond_κ holds for all regular κ , but \diamond_κ^* , \diamond_κ^+ hold exactly for those κ which are regular but not ineffable (see Kunen [5]).

The major application of $\diamond (= \diamond_{\omega_1})$ in set theory is to the construction of a Souslin tree in L . For this we refer the reader to Devlin [3]. In recursion theory an effectivized version of \diamond^* has been used in α -recursion theory. We conclude this lecture by sketching this latter application.

Given any notion of RE-ness a standard question to consider is Post’s Problem: Do there exist RE sets of incomparable degree? Often the priority method is used. The typical set-up is that one wishes to recursively enumerate sets A, B so as to satisfy requirements R_0, R_1, \dots , which are designed to guarantee that A, B are of incomparable degree. One hopes that each proper initial segment of requirements is permanently satisfied beyond some stage in the construction. In general,

the ordering of stages may have ordertype α greater than the length ρ of the listing of requirements. If $\text{cofinality}(\alpha) \geq \rho$ then the above hope is fulfilled.

If $\text{cofinality}(\alpha) < \rho$ then what is needed is a method by which requirement R_i can "guess" at the activity of the higher priority requirements $\langle R_j | j < i \rangle$. In case $\rho = \omega_1$ this "guess" can be provided by a \diamond^* -sequence $\langle D_\gamma | \gamma \in \omega_1 \rangle$, so that R_i uses D_i to lay out countably many possibilities. Fodor's Theorem is also relevant as the function $(i \rightarrow \text{least } j \text{ s.t. } R_j \text{ injures } R_i)$ is regressive.

It is worthwhile to point out that in the recursion-theoretic setting \diamond^* , and not \diamond , is the appropriate principle to adapt. It is typical of fine structure applications to recursion theory that combinatorial principles cannot be applied directly, but must be tailor-made for the problem at hand.

LECTURE 2. THE BOX PRINCIPLE AND MASTER CODES

The full flavor of the fine structure theory becomes first apparent through consideration of Jensen's Box Principle (\square). It is here as well that a recursion-theoretic intuition begins to play an important role.

If λ is a limit ordinal of cardinality $\kappa < \lambda$, then there exists a closed unbounded (CUB) set $C_\lambda \subseteq \lambda$ of ordertype at most κ . Such a subset of λ is called a *cofinalization* of λ . The Box Principle asserts that such C_λ 's can be chosen so as to cohere nicely for many different λ . For any $C \subseteq \text{ORD}$, $\text{Lim}(C)$ denotes the set of ordinals which are limits of elements of C .

$\square(\kappa)$: There exists a sequence $\langle C_\lambda | \kappa < \lambda < \kappa^+, \lambda \text{ limit} \rangle$ such that for all λ

- (a) C_λ is a closed unbounded subset of λ of ordertype $\leq \kappa$,
- (b) $\lambda' \in \text{Lim}(C_\lambda) \rightarrow C_{\lambda'} = C_\lambda \cap \lambda'$.

Note that we do not have *perfect coherence*: $\lambda' \in C_\lambda \rightarrow C_\lambda \cap \lambda' = C_{\lambda'}$. Perfect coherence contradicts (a) as it implies that $C_{\lambda'}$ is bounded in λ' when λ' is a successor element of C_λ . However, perfect coherence can be required if one will allow C_λ to be bounded in λ when $\text{cofinality}(\lambda) = \omega$.

We now proceed to describe the main points of Jensen's proof of $\square(\kappa)$ in L . It is convenient to work with a slightly modified version of $\square(\kappa)$, which we call $\square'(\kappa)$. Assume $V = L$ and let $S = \{\lambda | \kappa < \lambda < \kappa^+, \lambda \text{ limit and } L_\lambda \models \kappa \text{ is the largest cardinal}\}$.

$\square'(\kappa)$: There exists a sequence $\langle C_\lambda | \lambda \in S \rangle$ s.t. for all $\lambda \in S$

- (a) C_λ is a closed unbounded subset of λ of ordertype $\leq \kappa$,
- (b) $\lambda' \in \text{Lim}(C_\lambda) \rightarrow \lambda' \in S$ and $C_{\lambda'} = C_\lambda \cap \lambda'$.

Thus essentially we have here a $\square(\kappa)$ -sequence based on the ordinals in S only. Using the fact that S is a CUB subset of κ^+ , it is not difficult to derive $\square(\kappa)$ from $\square'(\kappa)$ in L .

Thus we will actually describe a proof of $\square'(\kappa)$. The proof will make use of a refinement of the Skolem hull operation $H_\lambda(X)$ from the preceding lecture. For any limit ordinal λ , $X \subseteq L_\lambda$ and $n \in \omega$, let $H_\lambda^n(X)$ consist of all $y \in L_\lambda$ which are Σ_n -definable in L_λ with parameters from X . Thus $H_\lambda(X) = \bigcup_n H_\lambda^n(X)$.

Now let $\nu \in S$; we wish to locate a "canonical" cofinalization C_ν of ν . To do so first consider $\beta(\nu) = \text{least } \beta \text{ s.t. some cofinalization of } \nu \text{ is } L_\beta\text{-definable}$. A more useful characterization of $\beta(\nu)$ is $\beta(\nu) = \text{least } \beta \text{ s.t. there is an } L_\beta\text{-definable function from a subset of } \kappa \text{ cofinally into } \nu$. Also, let $n(\nu) = \text{least } n \text{ s.t. some function from a subset of } \kappa \text{ cofinally into } \nu \text{ is } \Sigma_n \text{ over } L_{\beta(\nu)}$. In all fine structure arguments the pair $(\beta(\nu), n(\nu))$ plays a fundamental role in determining what happens at ν .

It is worthwhile to first consider the case $\beta(\nu) = \nu, n(\nu) = 1$. In this situation we can provide a natural recursion-theoretic definition of C . The key idea is to define what it means for a Σ_1 sentence ϕ with parameters from L_ν to be *true at stage* σ , where $\sigma < \nu$. This holds if the parameters in ϕ belong to L_σ and $L_\sigma \models \phi$. The essential feature of Σ_1 statements is their persistence: $\sigma_1 < \sigma_2 < \nu, \phi$ true at $\sigma_1 \rightarrow \phi$ true at σ_2 .

We are given the existence of a $\Sigma_1(L_\nu)$ function g from a subset of κ cofinally into ν . We can pick a parameter $p \in L_\nu$ so that some Σ_1 formula with parameter p defines g over L_ν . Thus for any γ, δ the statement " $g(\gamma) = \delta$ " is Σ_1 with parameters γ, δ, p and, therefore, we have assigned meaning to the assertion: $g(\gamma) = \delta$ is true at stage σ . A natural cofinalization of ν can be described as follows:

$$\nu_0 = 0, \quad \gamma_0 = \text{least } \gamma \in \text{Dom}(g) \text{ s.t. } g(\gamma) > 0, \quad \delta_0 = g(\gamma_0).$$

For $i > 0$

$$\nu_i = \text{least stage } \sigma \text{ ("} g(\gamma_j) = \delta_j \text{" is true at stage } \sigma \text{ for all } j < i),$$

$$\gamma_i = \text{least } \gamma \in \text{Dom}(g) \text{ s.t. } g(\gamma) > \nu_i,$$

$$\delta_i = g(\gamma_i).$$

Note that the sequence $\nu_0 < \nu_1 < \dots$ is continuous and increasing. The sequence $\gamma_0 < \gamma_1 < \dots$ is increasing. If ν_i is defined, so is γ_i and therefore so is ν_{i+1} . Thus $i_0 = \text{least } i \text{ so that } \nu_i \text{ is not defined}$ is a limit ordinal $\leq \kappa$ and the sequence $\langle \nu_i | i < i_0 \rangle$ is cofinal in ν . Most importantly, if λ is a limit ordinal less than i_0 , then the sequences $\langle \nu_i | i < \lambda \rangle, \langle \gamma_i | i < \lambda \rangle$ have the same definition over L_{ν_λ} as they do in L_ν . This is precisely the type of coherence property that we are looking for.

It is tempting to define C_ν to be $\{\nu_i | i < i_0\}$. The only difficulty is that what we have done depends on our choice of g . A final step (which will not be provided here) is required to make a canonical choice for g . The idea for doing this is to construe $H_\nu^1(\kappa \cup \{p\})$ as the range of some $\Sigma_1(L_\nu)$ function with domain contained in κ , for the least p such that $H_\nu^1(\kappa \cup \{p\})$ is unbounded in ν .

This completes the definition of C_ν in the case $\nu = \beta(\nu), n(\nu) = 1$. When $\beta(\nu) > \nu, n(\nu) = 1$, a natural modification of the above argument can be used to define C_ν . The main point is that the phrase " ϕ is true at stage σ " when ϕ is Σ_1 makes equally good sense for $L_{\beta(\nu)}$ as it did earlier for L_ν .

But what if $n(\nu) > 1$? We have now come to a crucial point in our discussion of the fine structure theory. The natural interpretation of Σ_1 predicates as being

enumerated in stages appears to break down when Σ_1 is replaced by Σ_n , $n > 1$. The essential problem is the lack of persistence: for $n > 1$ a Σ_n sentence true in L_σ need not be true in $L_{\sigma'}$ for all $\sigma' > \sigma$. The notion of Σ_{n-1} Master Code is designed to deal with this difficulty. Using it, Jensen showed that a Σ_n property can often be viewed as a property which is Σ_1 "relative" to a Σ_{n-1} -definable predicate. It is then possible to extend our earlier recursion-theoretic interpretation from Σ_1 to Σ_n , $n > 1$, thereby providing the missing ingredient needed to complete the proof of $\square'(\kappa)$.

We should explain what is meant by "relativization". Fix an ordinal β and let $A \subseteq L_\beta$. A formula is Σ_1^A if it is obtained from a Σ_1 formula by replacing some free variable by A . Now $B \subseteq L_\beta$ is Σ_1 relative to A if B is definable over the structure $\langle L_\beta[A], \epsilon, A \rangle$ by a Σ_1^A formula. The most manageable situation is where $L_\beta[A] = L_\beta$, in which case we say that $\langle L_\beta, A \rangle$ is amenable.

Fix $n = m + 1 > 1$ with a view toward obtaining a $\Sigma_m(L_\beta)$ -definable $A \subseteq L_\beta$ such that (a) any $B \subseteq L_\beta$ which is $\Sigma_{m+1}(L_\beta)$ is Σ_1 relative to A , and conversely, (b) any $B \subseteq L_\beta$ which is Σ_1 relative to A is $\Sigma_{m+1}(L_\beta)$. Property (a) is easy to arrange (at least for limit β): choose A to be any universal Σ_m predicate for L_β . However, (b) is much more difficult to obtain. It would help to at least choose A so that $\langle L_\beta, A \rangle$ is amenable. Then it can be shown that (b) reduces to: If $B = \{x \in L_\beta \mid x \subseteq A\}$ then B is $\Sigma_{m+1}(L_\beta)$. Even this is a problem unless we know that $\langle L_\beta, A \rangle$ is amenable for all $\Sigma_m(L_\beta)$ sets A , a property which fails for many β .

Jensen deals with this problem by working not with β itself but with its " Σ_m projectum". This is defined to be $\rho_m^\beta = \text{least } \rho \text{ s.t. there is a } \Sigma_m(L_\beta) \text{ injection from } \beta \text{ into } \rho$. We now work with $\Sigma_m(L_\beta)$ subsets of $\rho_m^\beta = \rho$ instead of $\Sigma_m(L_\beta)$ subsets of β . Thus we want $A \subseteq L_\rho$ such that (a') any $B \subseteq L_\rho$ which is $\Sigma_{m+1}(L_\beta)$ is Σ_1 relative to A and (b') any $B \subseteq L_\rho$ which is Σ_1 relative to A is $\Sigma_{m+1}(L_\beta)$.

A subset of $L_{\rho_m^\beta}$ obeying (a'), (b') is called a Σ_m Master Code for β . Jensen's fundamental Uniformization Theorem implies that $\langle L_\rho, A \rangle$ is amenable for all $\Sigma_m(L_\beta) A \subseteq L_\rho$; consequently, our earlier obstacles to demonstrating (b) have vanished and thus (b') can be proved. To obtain (a') let g be a $\Sigma_m(L_\beta)$ injection from L_β into ρ and define A to be the range of g on a universal Σ_m predicate for L_β . Thus a Σ_m Master Code for β exists.

Now let us return to our proof of $\square'(\kappa)$ and reconsider the case $n(\nu) = m + 1 > 1$. Thus, by definition, there is a $\Sigma_{m+1}(L_{\beta(\nu)})$ function g from a subset of κ cofinally into ν . Now view g instead as a function which is Σ_1 relative to some Σ_m Master Code A for $\beta(\nu)$. (This is possible as the leastness of $n(\nu)$ implies that $\nu \leq \rho_m^{\beta(\nu)}$ and, therefore, property (a') applies with $B = \text{Graph}(g)$.) It is now possible to define C_ν recursion-theoretically as before, this time over the structure $\mathcal{A}(\nu) = \langle L_{\rho_{n(\nu)-1}^{\beta(\nu)}}, A \rangle$. The only new point is that A must in fact be a canonical Σ_m Master Code for $\beta(\nu)$ to guarantee the coherence property of $\square'(\kappa)$. The structure $\mathcal{A}(\nu)$, when A is chosen to be the canonical $\Sigma_{n(\nu)-1}$ Master Code for $\beta(\nu)$, plays a major role in all fine structure arguments.

Lastly, the theory of Master Codes has had an impact on recursion theory. Assume $V = L$ and let κ be a cardinal. If $\kappa < \beta < \kappa^+$ and $\rho_n^\beta = \kappa$, then the canonical Σ_n Master Code for β is a subset of κ . The κ -degrees of these Master Codes are well ordered just as these Master Codes themselves are well ordered by $<_L$. In this way one obtains a κ -jump hierarchy $0 < 0' < 0'' < \dots$ cofinal in the κ -degrees. When $\kappa = \omega$, Jockusch and Simpson [6] and Hodes [7] have shown that the Turing degree 0^λ can often be degree-theoretically characterized in terms of $\{0^\alpha \mid \alpha < \lambda\}$. In case $\kappa = \aleph_{\omega_1}$ we have the following *Master Code Theorem*: Every κ -degree $\geq 0'$ is of the form 0^α for some α . This result is useful in the study of uncountable admissible ordinals.

LECTURE 3. STRONG CODING

In this lecture we apply fine structure theory to the study of the Admissibility Spectrum.

An ordinal $\alpha > \omega$ is *admissible* if L_α is a model of the Σ_1 -Replacement scheme, obtained from the usual Replacement scheme by restricting it to Σ_1 formulas. For any $A \subseteq \text{ORD}$, α is *A-admissible* or *admissible relative to A* if $\langle L_\alpha[A], \varepsilon, A \cap \alpha \rangle$ obeys Σ_1^A -Replacement. (Note that if A is a bounded subset of α , then this reduces to $L_\alpha(A)$ is a model of Σ_1 -Replacement.) The *Admissibility Spectrum* of A is the class of all A -admissible ordinals, denoted by $\Lambda(A)$. We also introduce the notation $\Lambda(\emptyset) = \Lambda$ and $\alpha(A) = \min(\Lambda(A))$.

Observations. (1) $\alpha(\emptyset) = \omega_1^{\text{ck}}$, the least nonrecursive ordinal. The notion of admissible ordinal arose from the generalization of recursion theory on ω_1^{ck} , metarecursion theory, to recursion theory on α , α -recursion theory.

(2) If $A \in L_\alpha$, α admissible, then α is A -admissible. The converse is false even for constructible subsets of ω ; indeed, Sacks showed that $\{R \subseteq \omega \mid \alpha(R) = \omega_1^{\text{ck}}\}$ has measure 1 in Cantor Space (provably in ZF).

(3) If α is admissible, $\mathcal{P} \in L_\alpha$ is a partial-ordering and G is \mathcal{P} -generic over L_α , then α is G -admissible.

The Admissibility Spectrum Problem is that of determining which Admissibility Spectra can occur. We shall focus here on the Admissibility Spectra of subsets of ω , or *reals*. This was studied by Sacks and by Jensen who showed:

(Sacks) Any countable admissible ordinal $> \omega$ is $\alpha(R)$ for some real R .

(Jensen) Suppose $\omega < \alpha_0 < \alpha_1 < \dots$ is a countable sequence of countable admissibles and for each i , α_i is admissible relative to $\{\alpha_j \mid j < i\}$. Then for some real R , $\{\alpha_0, \alpha_1, \dots\}$ is an initial segment of $\Lambda(R)$.

Note that by (2), if $R \in L$ then $\Lambda(R)$ must agree with Λ beyond some L -countable ordinal. Moreover, (3) implies that if R belongs to a set-generic extension of L then $\Lambda(R)$ agrees with Λ beyond *some* ordinal.

Are other Admissibility Spectra possible? Yes, for $\Lambda(0\#)$ is contained in the L -cardinals. But are large cardinals necessary? This led Solovay to pose the following:

Solovay's Question. Is it consistent that for some real R , $\Lambda(R) = \text{RI} = \text{The Recursively Inaccessible Ordinals}$? (α is *recursively inaccessible* if α is admissible and the limit of admissible ordinals.)

The theorem that we wish to discuss is

THEOREM. $\text{Con ZF} \rightarrow \text{Con}(\text{ZF} + \exists R \subseteq \omega(\Lambda(R) = \text{RI}))$. Specifically there is a class-generic extension N of the minimal model of ZF and a real $R \in N$ s.t. $N \models \Lambda(R) = \text{RI}$.

The proof of this theorem is based on a refinement of the technique of Jensen Coding which we call "Strong Coding". It is worthwhile to first consider the following weakened version of the Jensen Coding Theorem, in order to illustrate Jensen's technique.

THEOREM (JENSEN). Suppose $A \subseteq \text{ORD}$ and $A \cap \alpha \in L$ for all α ($\langle L, A \rangle$ is amenable). Then there is a class-generic extension M of L s.t. $M \models \text{ZF}$ and for some real $R \in M$: $M = L(R)$, A is definable over $L(R)$.

We will now give a very rough description of Jensen's proof. First consider the simpler problem of "coding" a subset of ω_1 by a real: Thus let $B \subseteq \omega_1$. We wish to devise a forcing notion \mathcal{R}_0^B s.t. if G is \mathcal{R}_0^B -generic then for some real R : $V[G] = V[R]$, B is definable in $V[R]$ from the parameter R . This can be done with "almost disjoint forcing", a method due to Solovay. We assume $V = L$. Using this hypothesis we can choose a "canonical" method of assigning a real r_ξ to each countable ordinal ξ so that if ξ_1, \dots, ξ_n are distinct then r_{ξ_1} is almost disjoint from $r_{\xi_2} \cup \dots \cup r_{\xi_n}$ ($a, b \subseteq \omega$ are *almost disjoint* if $a \cap b$ is finite). Then our forcing \mathcal{R}_0^B is defined so as to produce a generic real R s.t. $\xi \in B \leftrightarrow R$ is almost disjoint from r_ξ . In this way R "codes" B . More specifically: A condition $p \in \mathcal{R}_0^B$ is a pair (s, \bar{s}) where s is a finite subset of ω and \bar{s} is a finite subset of $\{r_\xi \mid \xi \in B\}$. And, (t, \bar{t}) extends (s, \bar{s}) if $t \supseteq s$, $\bar{t} \supseteq \bar{s}$, all elements of $t - s$ are greater than $\max(s)$ and $(t - s) \cap r_\xi = \emptyset$ for $r_\xi \in \bar{s}$. It is easy to check that if G is \mathcal{R}_0^B -generic then $R_G = \bigcup \{s \mid (s, \bar{s}) \in G\}$ has the desired property. Moreover, \mathcal{R}_0^B has the countable chain condition.

The idea for proving Jensen's Theorem is to reason as follows: The forcing $\mathcal{R}_0^{A \cap \omega_1}$ provides us with a "canonical procedure" for coding $A \cap \omega_1$ by a real. By generalizing almost disjoint forcing one cardinal higher, we can similarly code $A \cap \omega_2$ by a subset of ω_1 , using an analogous forcing $\mathcal{R}_1^{A \cap \omega_2}$. But then by combining these two forcings we have a method for coding $A \cap \omega_2$ into a real. Jensen coding allows one to "iterate" this procedure through all the cardinals so as to code A by a real; an appropriate modification of almost disjoint forcing is needed at limit cardinals. The fine structure theory ($\diamond, \square, \text{gap-1 morass}$) is needed for the proof.

Now the above ideas suggest the following approach to Solovay's problem ($\Lambda(R) = \text{RI}$):

(1) Find $A \subseteq \text{ORD}$ such that $\langle L, A \rangle$ is amenable and $\Lambda(A) = \text{RI}$.

(2) Code A by a real R in such a way that $\Lambda(R) = \Lambda(A)$.

(1) is easy to arrange by choosing canonical cofinalizations of each successor admissible and then putting them together into a single predicate A (in fact, A is Δ_1 -definable). The problem is with (2). What we want to arrange is

(2a) $A \cap \alpha$ is Δ_1 over $L_\alpha(R)$ for every admissible α ,

(2b) α A -admissible $\rightarrow \alpha$ R -admissible.

The property (2a) guarantees that $\Lambda(R) \subseteq \Lambda(A)$ and (2b) states that $\Lambda(A) \subseteq \Lambda(R)$. Intuitively, (2a) states that the "decoding" of A from R is so efficient that it can be carried out inside $L_\alpha(R)$ for every admissible α .

Unfortunately the Jensen Coding method does not provide this. For, to determine whether or not ξ belongs to $A \cap \omega_1$, we must first determine the "code" r_ξ and then ask if R is almost disjoint from r_ξ . However, not every countable admissible α will be closed under the operation $\xi \mapsto r_\xi$, as in general, r_ξ will not appear in L until a level much greater than ξ . In fact, the best that can be done is to obtain r_ξ inside L_{μ_ξ} where $\mu_\xi = \text{least } \beta \text{ s.t. } L_\beta \models \xi \text{ is countable}$. As a result the best that we can appear to obtain is: If α is a countable admissible then $A \cap (\omega_1)^{L_\alpha}$ is Δ_1 over $L_\alpha(R)$.

However, the following trick enables us to get around this problem and thereby establish (2a): Note that we would have what we want if $A \cap \alpha$ were Δ_1 over $L_\alpha[A \cap (\omega_1)^{L_\alpha}]$. The idea now is to apply the coding technique not to A but to a predicate A' which has the preceding property and is such that A is simply coded into A' (say $A = \text{even part of } A'$). Then A' , and hence A , can be "decoded" from the generic real R at every admissible ordinal. In actual fact, the predicate A' must be built generically and "simultaneously" with the generic real coding it. (The assumption that A is Σ_1 is needed here.) In this way we get (2a) and hence the partial result

(*) $\text{Con ZF} \rightarrow \text{Con}(\text{ZF} + \exists R(\Lambda(R) \subseteq \text{RI}))$.

This result was obtained independently by René David [10] who also showed using a similar approach that RI can be replaced in (*) by any Σ_1 class of admissibles $X \supseteq L$ -cardinals.

Notice however that by passing from A to A' we may have destroyed the admissibility of some of the A -admissible ordinals. In other words, $\Lambda(A')$ may be a proper subset of $\Lambda(A)$. The purpose of the Strong Coding method is to remove this defect. Once again we need to "improve" A to a predicate A' with the property that for any admissible α of cardinality $\kappa < \alpha$: $A' \cap \alpha$ is Δ_1 over $L_\alpha[A' \cap (\kappa^+)^{L_\alpha}]$. But now we want to also guarantee that the admissibility of any A -admissible ordinal is preserved by A' .

Note that the problem we are dealing with can itself be viewed as a "coding" problem: in this case we want to build $A' \cap (\kappa^+)^{L_\alpha}$ so as to code (at least) $A \cap \alpha$.

This suggests the solution: We should make $A' \cap (\kappa^+)^{L_\alpha}$ itself generic for a Jensen-style coding of the "universe" $\langle L_\alpha[A], \varepsilon, A \cap \alpha \rangle$ by a subset of $(\kappa^+)^{L_\alpha}$. Thus the desired forcing for coding the predicate A by a real should be constructed from conditions which are themselves generic solutions of the problem of coding initial segments $A \cap \alpha$ of A .

More explicitly, let $\text{Adm} = \{\beta \mid \beta \text{ is admissible or the limit of admissibles}\}$ and for $\beta \in \text{Adm}$, $\beta\text{-Card} = \{\kappa \mid \kappa = 0 \text{ or } \kappa \text{ is an infinite } \beta\text{-cardinal}\}$. Let $0^+ = \omega$. For each $\beta \in \text{Adm}$, $\kappa \in \beta\text{-Card}$, a forcing \mathcal{P}_κ^β is defined so as to produce a subset of $(\kappa^+)^{L_\beta}$ which codes $A \cap \beta$. A condition $p \in \mathcal{P}_\kappa^\beta$ is typically a function on an initial segment of $(\beta\text{-Card}) - \kappa$ which assigns a pair $p(\delta) = (p_\delta, \bar{p}_\delta)$ to each $\delta \in \text{Dom}(p)$. The pair $(p_\delta, \bar{p}_\delta)$ is a condition in the forcing \mathcal{P}^{p_δ} for coding $A \cap \delta^+$, p_δ into a subset of δ^+ (δ^+ denotes $(\delta^+)^{L_\beta}$). There are also requirements at limit β -cardinals as well which will not be discussed here. Finally, the set $p_\delta \cap (\delta^+)^{L_\alpha}$ must be a $\mathcal{P}_\delta^\alpha$ -generic coding of $p_\delta \cap \alpha$, for every admissible α between δ and $\text{sup}(p_\delta)$.

Thus our conditions are much like those used in Jensen Coding, but with the added restriction that the "coding elements" p_δ be themselves generic for (strong) codings at smaller ordinals. This restriction complicates the proofs of the basic lemmas from Jensen's argument; in particular, more fine structure is needed. There are two principal difficulties to handle, which we now describe.

The first is raised by the question: Do nontrivial conditions exist? To provide a positive answer one must build generic sets in L for forcings \mathcal{P}_κ^β where κ is uncountable, $\text{card}(\beta) = \kappa$. If we knew that both L_β and \mathcal{P}_κ^β were $< \kappa$ -closed, then this would of course be easily done, but this is not generally the case. However, \mathcal{P}_κ^β can be shown to be κ -distributive in L_β ; thus one could hope to get a \mathcal{P}_κ^β -generic set by decomposing L_β into pieces $\langle B_i \mid i < i_0 \rangle$, $i_0 \leq \kappa$ and then defining a sequence of conditions $p_0 \geq p_1 \geq \dots$ where p_i meets all dense sets in B_i , each $B_i \in L_\beta$ having cardinality $\leq \kappa$. This approach works, provided the above type of decomposition can be obtained. It cannot in general, but instead a series of decompositions can be used, as defined by the *critical projecta* of β .

For any limit ordinal $\nu \notin \text{Card}$, define $\beta(\nu) = \text{least } \beta \text{ s.t. there is an } L_{\beta\text{-definable}} f: \nu \rightarrow^{1-1} \nu'$, some $\nu' < \nu$. Also, $n(\nu) = \text{least } n \text{ s.t. such an } f \text{ is } \Sigma_n(L_{\beta(\nu)})$, $\rho(\nu) = (\Sigma_{n(\nu)}\text{-projectum of } \beta(\nu)) < \nu$ and $\rho'(\nu) = \Sigma_{n(\nu)-1}\text{-projectum } (\beta(\nu)) \geq \nu$. Define the sequence $\rho_0 = \rho'_0 = \nu$; $\rho_{i+1} = \rho(\rho_i)$, $\rho'_{i+1} = \rho'(\rho_i)$; \dots . The ρ_i 's are descending so for some least $k(\nu)$, $\rho_{k(\nu)} = \text{card}(\nu)$. The *principal projecta* of ν are the ρ_i 's, the *auxiliary projecta* of ν are the ρ'_i 's and the *critical projecta* of ν are both. Thus we are describing the way in which ν becomes collapsed in L . For the sake of the present discussion, we deal only with the principal projecta.

Suppose now that $\kappa = \text{card}(\beta)$ and we wish to obtain a \mathcal{P}_κ^β -generic set G . Let $\beta = \rho_0 > \rho_1 > \dots > \rho_k = \kappa$ be the principal projecta of β . G is constructed in k steps: First build a $\mathcal{P}_{\rho_1}^\beta$ -generic set G_1 , then a $\mathcal{P}_{\rho_2}^{G_1}$ -generic G_2, \dots , a $\mathcal{P}_\kappa^{G_{k-1}}$ -generic G_k . Here, $\mathcal{P}_{\rho_i}^{G_{i-1}}$ denotes a forcing designed to strongly code $A \cap \rho_i^+$, G_{i-1} by a subset of ρ_i^+ . The desired G is obtained by "glueing together" the G_i 's: $p \in G$ iff

$p \upharpoonright (\text{Dom}(p) - \rho_i) \in G_i$. Our success in obtaining G_i is based on the fact that L_β can be decomposed into an i -dimensional array of size $\leq \rho_i$, consisting of elements of L_β of size $\leq \rho_i$. This array is constructed using $\square(\rho_1), \square(\rho_2), \dots, \square(\rho_i)$.

The second difficulty is that the usual type of almost disjoint forcing does not mix well with the genericity restriction that we have placed on the p_δ 's. Specifically, suppose $(p_\delta, \bar{p}_\delta)$ is a condition for coding $A \cap \delta^+, p_{\delta^+}$ into a subset of δ^+ and we wish to extend $(p_\delta, \bar{p}_\delta)$ to a condition $(q_\delta, \bar{q}_\delta)$, $\text{sup}(q_\delta) = \alpha$. We may assume that $\alpha > \text{sup}(p_\delta)$ is admissible. Our restriction says that q_δ must be generic for $\mathcal{P}_\delta^\alpha$, yet the almost disjoint forcing also requires $q_\delta - p_\delta$ is disjoint from $\cup\{r_\xi \mid r_\xi \in \bar{p}_\delta\}$. We must justify the compatibility of these two requirements. Doing this requires that the restrictions of the codes $\{r_\xi \cap (\delta^+)^{L_\alpha} \mid r_\xi \in \bar{p}_\delta\}$ be generic for some forcing defined over L_α .

Thus the r_ξ 's must be chosen so that $r_\xi \cap \alpha$ is "generic" for every admissible $\alpha < \delta^+$. Cohen genericity cannot be used as if r_ξ were Cohen generic then $r_\xi \cap [\alpha_1, \alpha_2] = \emptyset$ for many large intervals $[\alpha_1, \alpha_2]$ below δ^+ , and thus $r_\xi \cap \alpha_2$ could not be generic. Instead both the r_ξ 's and the forcings for which the r_ξ 's are generic must be defined by induction on a gap-1 morass at δ^+ . $\square(\delta)$ is needed to get through limit stages of this induction.

This completes our outline of the Strong Coding technique. It is our hope that these lectures have helped to suggest a fruitful interplay between recursion theory and ideas in the fine structure of L .

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