Cantor's Set Theory from a Modern Point of View

Georg Ferdinand Ludwig Philipp Cantor

Berlin doctorate 1867 (number theory)
Halle habilitation 1870 (number theory)
Heine $\rightarrow$ Study of trigonometric series $\rightarrow$
Set theory

Theory of transfinite numbers and cardinality
Algebraic numbers are countable
Real numbers are not countable
1-1 correspondence between n-dimensional space and the real line

Halle Chair 1879
Founder of the DMV 1890

Opposition from Kronecker
Support from Dedekind
Mittag-Leffler: "100 years too soon"

Transfinite counting

C closed set of reals

$C' =$ limit points of C (Cantor derivative)

$C \supseteq C' \supseteq C'' \supseteq \cdots$

$C^\infty =$ the intersection

$C^\infty \supseteq (C^\infty)'$, maybe strict!

Keep counting: $C^\infty \supseteq C^{\infty+1} \supseteq C^{\infty+2} \supseteq \cdots$!

What is $0, 1, ..., \infty, \infty + 1, ...$?

Wellordering: Linear ordering with no infinite descending sequence

Cantor: Any 2 wellorderings are comparable

Each wellordering isomorphic to an *ordinal*, a special wellordering ordered by $\in$

$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, ..., \omega = \{0, 1, 2, ...\}$,
$\omega + 1 = \omega \cup \{\omega\}, \ldots$

Cantor's assumption: Every set can be wellordered
Therefore every set bijective with an ordinal (not unique)

Cardinal = Ordinal not bijective with a smaller ordinal
Every set bijective with a *unique* cardinal, its *cardinality*

Zermelo: Cantor's assumption follows from the Axiom of Choice
So Cantor's theory of cardinality applies to arbitrary sets

One major gap!

What is the cardinality of the continuum?

Continuum Hypothesis (CH):
Every uncountable set of reals has the same cardinality as the set of all reals

Paradoxes

Cantor, Burali-Forti, Russell
$x =$ all y such that $y \notin y$
$x \in x \leftrightarrow x \notin x$!

Zermelo's proposal
Only use established principles of set-formation
Axiomatic theory: Zermelo set theory
ZFC = Zermelo-Fraenkel set theory with the Axiom of Choice

The Universe of Sets V

ZFC gives the following picture:

First picture of V
Reduces V to ordinals and power set operation
Not a clear description

The Vagueness of Power Set

Approach 1

Definable sets: descriptive set-theory
Borel sets = smallest σ-algebra containing all open sets

Σ^1_1 = continuous image of a Borel set
Π^1_1 set = complement of Σ^1_1 set
Σ^1_{n+1} set = continuous image of Π^1_n set
Π^1_{n+1} set = complement of Σ^1_{n+1} set
Projective = Σ^1_n or Π^1_n for some n

1930s
Σ^1_1 sets satisfy CH: an uncountable Σ^1_1 set has the cardinality of the reals
Π^1_1 sets?

Approach 2

Constructibility (Gödel, late 1930's)
Replace power set operation by a weak power set operation:
$V_{\alpha+1} =$ all subsets of V_α
$L_{\alpha+1} =$ all "simple" subsets of L_α
$L =$ union of the L_α's
L satisfies ZFC
First clearly-described model of ZFC
CH holds in L!

Gödel:
L is *not* the correct intepretation of ZFC
Only a tool for showing that statements are consistent with ZFC

There are other interpretations of ZFC:

Cohen's Forcing method
Add new sets to L, preserving ZFC
R is *Cohen over* L iff
R belongs to every open dense set of reals which L can "describe"
Add many Cohen reals to L, obtain model where CH fails

Another use of forcing: R in $[0, 1]$ is *random over* L iff
R belongs to every measure one subset of $[0, 1]$ which L can "describe"
Using random reals: Model where every projective set of reals is Lebesgue measurable

Thus *CH* and *Projective sets measurable* are undecidable using the ZFC axioms

Dilemma: Different universes with different kinds of mathematics?

Why not $V = L$*?*

Con(T): T is consistent

Gödel's 2nd Incompleteness Theorem:
T does not prove Con(T)!

Con(ZFC+CH) $\leftrightarrow$ Con(ZFC+ $\sim$ CH) $\leftrightarrow$ Con(ZFC)

Con(ZFC $+ V = L$) $\leftrightarrow$ Con(ZFC)

But:

ZFC+ Projective sets measurable $\rightarrow$ Con(ZFC)

and therefore by Gödel:

Con(ZFC) $\nrightarrow$
Con(ZFC+ Projective sets measurable)

ZFC $+ V = L$ is too weak for proving consistency

How do we extend ZFC to make it strong for consistency?

Example from measure theory
Countably additive extension of Lebesgue measure to all sets of reals $\rightarrow V$ is not L

Model of ZFC with such a measure $\leftrightarrow$
Model of ZFC with a *measurable cardinal*

Measurable cardinal: example of a "large cardinal hypothesis"
These hypotheses have a crucial role in set theory:

φ is *consistency-equivalent* to ψ:
$\mathrm{Con}(\mathrm{ZFC} + \varphi) \leftrightarrow \mathrm{Con}(\mathrm{ZFC} + \psi)$
Empirical fact:
For any natural set-theoretic assertion φ, φ is consistency-equivalent to $0 = 0$, $0 = 1$ or to a large cardinal hypothesis
Large cardinal hypotheses measure the consistency strength of set-theoretic assertions

More than a measurable cardinal is needed to measure strength:

A is *Wadge reducible* to B iff
For some continuous f, $x \in A$ iff $f(x) \in B$

WP_n: If A, B are Σ^1_n but not Π^1_n then
A is Wadge reducible to B and vice-versa

We have:

WP_1 is consistency equivalent to #'s, a large cardinal hypothesis below a measurable cardinal.
WP_2 is consistency equivalent to the existence of a Woodin cardinal, much larger than a measurable cardinal!
WP_n requires $n-1$ Woodin cardinals

Why should Woodin cardinals be consistent? Maybe WP_n is simply false for $n > 1$!

Gödel again

Maximum principles: The universe V is large

Gödel:

"I believe that the basic problems of abstract set theory, such as Cantor's continuum problem, will be solved satisfactorily only with the help of axioms of *this* kind."

The inner model hypothesis
If a sentence holds in an inner model of some outer model of V (i.e., in some model compatible with V), then it already holds in some inner model of V.

The IMH implies that there are no inaccessible cardinals in V

The IMH implies however that there are measurable cardinals in inner models

The IMH is consistent relative to Woodin cardinals

The strong inner model hypothesis
If a sentence with an absolute parameter p holds in an inner model of some outer model of V which respects the size of p, then it already holds in some inner model of V.

The SIMH solves the continuum problem negatively

The SIMH implies that there are strong cardinals in inner models

Q: Is the SIMH consistent relative to large cardinals?

In summary:

1. Cantor's set theory was highly successful, but suffered from paradoxes and left CH unresolved.
2. The paradoxes were resolved by the development of axiomatic set theory, ZFC.
3. Gödel revealed the weakness of ZFC for proving consistency.
4. Gödel and Cohen showed that ZFC does not resolve CH.
5. Large cardinals resolve the consistency weakness of ZFC.
6. Maximum principles show promise both for justifying the consistency of large cardinals and for resolving CH.

Will set theory reach a definitive picture of the universe of sets?

Time will tell ...