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The enriched stable core and the relative rigidity of HOD

by

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Abstract. In the author's 2012 paper, the V-definable Stable Core $\mathbb{S} = (L[S], S)$ was introduced. It was shown that V is generic over $\mathbb{S}$ (for $\mathbb{S}$-definable dense classes), each V-definable club contains an $\mathbb{S}$-definable club, and the same holds with $\mathbb{S}$ replaced by (HOD, S), where HOD denotes Gödel's inner model of hereditarily ordinal-definable sets. In the present article we extend this to models of class theory by introducing the V-definable *Enriched Stable Core* $\mathbb{S}^* = (L[S^*], S^*)$. As an application we obtain the rigidity of $\mathbb{S}^*$ for all embeddings which are "constructible from V". Moreover, any "V-constructible" club contains an "$\mathbb{S}^*$-constructible" club. This also applies to the model (HOD, S^*), and therefore we conclude that, relative to a V-definable predicate, HOD is rigid for V-constructible embeddings.

In this article we introduce the *Enriched Stable Core*, a generalisation of the Stable Core of [2], and use it to study the rigidity of HOD, Gödel's universe of hereditarily ordinal-definable sets. We begin with a review of the Stable Core (taking the opportunity to correct an error in the presentation of [2]).

For an infinite cardinal α, $H(\alpha)$ consists of those sets whose transitive closures have size less than α. Let C denote the closed unbounded class of all infinite cardinals β such that $H(\alpha)$ has cardinality less than β whenever α is an infinite cardinal less than β.

Definition 1. For a finite $n > 0$, we say that α is *n-Admissible* if α is a limit point of C and $(H(\alpha), C \cap \alpha)$ satisfies Σ_n replacement (with $C \cap \alpha$ as an additional unary predicate). We say that α is *n-Stable in β* if $\alpha < \beta$ and $(H(\alpha), C \cap \alpha)$ is Σ_n-elementary in $(H(\beta), C \cap \beta)$.

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The *Stability predicate* S consists of all triples (α, β, n) such that α is n-Stable in β and β is n-Admissible ([1]). The Δ_2-definable predicate S describes the "core" of V, in the following sense.

THEOREM 2. *V is generic over $(L[S], S)$ for an $(L[S], S)$-definable forcing. The same is true with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model M.*

COROLLARY 3. *V is generic over* HOD *via a forcing which is definable in V.*

In general, the inner model $L[S]$ may be strictly smaller than HOD; it also obeys more absoluteness than that exhibited by HOD. See [2] for more about this.

The proof of Theorem 2 comes in two parts. First it is shown that V can be written as $L[F]$ where F is a function from the ordinals to 2 which "preserves" the Stability predicate S, in the sense that if α is n-Stable in β and β is n-Admissible then α is also n-Stable in β relative to F ([2]). Then the function F is used to prove the genericity of V over $(M[S], S)$ for any definable inner model M.

To obtain F we first define by induction on $\beta \in C$ a collection $\mathbb{P}(\beta)$ of functions from β to 2. If β is not a limit point of C then $\mathbb{P}(\beta)$ consists of all functions $p : \beta \to 2$ such that $p\restriction\alpha$ belongs to $\mathbb{P}(\alpha)$ for all $\alpha \in C \cap \beta$. Suppose that β is a limit point of C and let $\mathbb{P}({<}\beta)$ denote the union of the $\mathbb{P}(\alpha)$, $\alpha \in C \cap \beta$, ordered by extension. Assuming *extendibility for* $\mathbb{P}({<}\beta)$, i.e. the statement that for $\alpha_0 < \alpha_1 < \beta$ in C, each q_0 in $\mathbb{P}(\alpha_0)$ can be extended to some q_1 in $\mathbb{P}(\alpha_1)$, this forcing adds a generic function which we denote by $\dot{f} : \beta \to 2$. We say that $p : \beta \to 2$ is *n-generic for* $\mathbb{P}({<}\beta)$ if $G(p) = \{p\restriction\alpha \mid \alpha \in C \cap \beta\}$ meets every dense subset of $\mathbb{P}({<}\beta)$ of the form $\{q \in \mathbb{P}({<}\beta) \mid q \Vdash \varphi \text{ or } q \Vdash {\sim}\varphi\}$, where φ is a $\Pi_n(H(\beta), C \cap \beta, \dot{f})$ sentence with parameters from $H(\beta)$. We then take $\mathbb{P}(\beta)$ to consist of all $p : \beta \to 2$ which are n-generic for $\mathbb{P}({<}\beta)$ for all n such that β is n-Admissible.

Let $\mathbb{P}$ be the union of all of the $\mathbb{P}(\beta)$'s, ordered by extension. The following are proved as Lemmas 7 and 6 in [2], respectively:

LEMMA 4. *Suppose that $\alpha < \beta$ belong to C and p belongs to $\mathbb{P}(\alpha)$. Then p has an extension q in $\mathbb{P}(\beta)$.*

LEMMA 5. *Suppose that G is $\mathbb{P}$-generic over V and let F be the union of the functions in G. Then $V = L[F]$ and F preserves the Stability predicate. Moreover, V satisfies replacement with F as an additional predicate.*

([1]) The requirement that β be n-Admissible was missing in [2].

([2]) We do not require that the n-Admissibility of β be preserved by F, although this could be achieved with a more complicated argument.

In [2] there was an error in the proof of Lemma 6 of that paper (which corresponds to Lemma 5 above): To obtain the "n-genericity of $F \restriction \beta$" on line 14 of that proof, one needs the n-Admissibility of β. To fix this we have now built n-Admissibility into the definition of the Stability predicate S.

To obtain the genericity of V over $(L[S], S)$, a forcing $\mathcal{Q}$ is defined consisting of sentences in an infinitary propositional logic (with arbitrary conjunctions and disjunctions in $L[S]$) which are consistent with a theory T which captures the n-Stability relationships specified by the predicate S (see [2, p. 265]; the new definition of S requires the added requirement that β be n-Admissible in clause (b) on that page). A similar but more complex argument is given in the proof of Theorem 10 below. The same argument works with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model M.

With these modifications of the treatment in [2], Theorem 2 is established.

Rigidity. As V is generic over the Stable Core $\mathbb{S} = (L[S], S)$ (where S is the Stability predicate) for a definable forcing whose definable antichains are sets, we obtain as a consequence:

COROLLARY 6. *Any V-definable club contains an $\mathbb{S}$-definable club. And $\mathbb{S}$ is rigid for V-definable embeddings, i.e., there is no V-definable elementary embedding of $\mathbb{S}$ to itself other than the identity.*

Proof. The statement about clubs follows immediately from the fact that V is generic over $\mathbb{S}$ for a definable forcing whose definable antichains are sets.

We give two proofs of rigidity for V-definable embeddings, as both are useful for generalisations, such as the second conclusion of Theorem 10 below. In that argument, the analogous first proof is simpler, however the analogous second proof can be applied to theories weaker than Morse–Kelley.

First proof. Suppose that V is $\mathbb{P}$-generic over $\mathbb{S}$ for the $\mathbb{S}$-definable forcing $\mathbb{P}$ and that there were an elementary (equivalently, Σ_1-elementary) embedding of $\mathbb{S}$ to itself which is Σ_n-definable over V. Let κ be the least ordinal which is forced to be the critical point of such a Σ_n-definable embedding by some condition in $\mathbb{P}$. Then κ is $\mathbb{S}$-definable and therefore cannot be moved by any elementary embedding from $\mathbb{S}$ to itself, a contradiction.

Second proof. We first claim that there is an $\mathbb{S}$-definable $\diamondsuit$-sequence for $\mathbb{S}$ that concentrates on ordinals of cofinality ω and guesses $\mathbb{S}$-definable classes on $\mathbb{S}$-definably stationary classes (3). More precisely, there is an $\mathbb{S}$-definable sequence $(X_\alpha \mid \alpha \text{ of } \mathbb{S}\text{-cofinality } \omega)$ such that $X_\alpha \subseteq \alpha$ for each α and whenever X is an $\mathbb{S}$-definable class of ordinals and D is an $\mathbb{S}$-definable club there is α in D such that $X \cap \alpha = X_\alpha$. To see this, define X_α inductively

(3) There is nothing special about the predicate S in this argument.

as follows: Let n be least such that some pair (X_α, D_α) is Σ_n-definable over $(L_\alpha[S], S \cap L_\alpha[S])$ and such that $X_\alpha \subseteq \alpha$, D_α is club in α and $X_\alpha \cap \bar{\alpha} \neq X_{\bar{\alpha}}$ for all $\bar{\alpha} \in D_\alpha$. If α does not have $\mathbb{S}$-cofinality ω or if there is no such pair, then we set $X_\alpha = \emptyset$; otherwise we let (X_α, D_α) be the least such pair (where Σ_n sets are ordered by the formulas which define them and for a fixed formula by the parameters used). We claim that the sequence $(X_\alpha \mid \alpha$ of $\mathbb{S}$-cofinality $\omega)$ is as desired. If not, let n be least such that some $X \subseteq \mathrm{Ord}$ which is Σ_n-definable over $\mathbb{S}$ is not guessed correctly anywhere on some Σ_n-definable club $D \subseteq \mathrm{Ord}$; fix the least such pair (X, D), and notice that by reflection there is an α of $\mathbb{S}$-cofinality ω such that $X \cap \alpha = X_\alpha$, $D \cap \alpha = D_\alpha$. But this is a contradiction because α belongs to D.

Now use the $\diamondsuit$-sequence to produce an $\mathbb{S}$-definable partition $(X_i \mid i \in \mathrm{Ord})$ of the ordinals of $\mathbb{S}$-cofinality ω into pieces which are $\mathbb{S}$-definably stationary (i.e. which intersect each $\mathbb{S}$-definable club). (For example, choose X_i to consist of those α of $\mathbb{S}$-cofinality ω such that $D_\alpha = \{i\}$ for $i > 0$, and X_0 to consist of the remaining α's.) Suppose that $j : \mathbb{S} \to \mathbb{S}$ were elementary with critical point κ with j definable in V. Now $D = \{\alpha \mid j[\alpha] \subseteq \alpha\}$ is a V-definable club and therefore contains an $\mathbb{S}$-definable club; it follows that there is an ordinal α of $\mathbb{S}$-cofinality ω in $j((X_i \mid i \in \mathrm{Ord}))_\kappa$ such that $j[\alpha] \subseteq \alpha$, and therefore $j(\alpha) = \alpha$. But then as $j(\alpha)$ belongs to $j((X_i \mid i \in \mathrm{Ord}))_i$ for some $i < j(\kappa)$, it follows that α belongs to X_i for some $i < \kappa$ and therefore $j(\alpha) = \alpha$ belongs to $j((X_i \mid i \in \mathrm{Ord}))_i$ for some $i < \kappa$; this contradicts the fact that $j((X_i \mid i \in \mathrm{Ord}))$ is a partition into disjoint pieces ([4]). ∎

But what about embeddings that are not V-definable?

From now on we work in Gödel–Bernays class theory, whose models look like $(V, \mathcal{C})$ where V consists of the sets and $\mathcal{C}$ consists of the classes. A reformulation of the previous corollary is:

Corollary 7. *Suppose $(V, \mathcal{C})$ is the least model of Gödel–Bernays built over V (i.e., $\mathcal{C}$ consists only of the V-definable classes). Also let $(L[S], \mathcal{C}^S)$ be the least model of Gödel–Bernays built over $L[S]$ which has S as a class (i.e. $\mathcal{C}^S$ consists only of the $\mathbb{S}$-definable classes). Then any club in $\mathcal{C}$ contains a club in $\mathcal{C}^S$, and $\mathbb{S}$ is rigid for embeddings in $\mathcal{C}$.*

To obtain rigidity of the Stable Core in larger models of Gödel–Bernays we put more information into the Stability predicate.

The Enriched Stable Core. We define the Enriched Stability predicate S^* as follows. For β in C, $i < \beta^+$ of $L(H(\beta))$ and $0 < n < \omega$ we say that β is (i, n)-*Admissible* if β is a limit point of C and β is regular with respect to functions which are $\Sigma_n(L_i(H(\beta)), C \cap \beta)$ with parameters

([4]) This argument traces back to Woodin's proof of Kunen's rigidity theorem (see [3]).

from $H(\beta) \cup \{H(\beta)\}$ (just $H(\beta)$ if $i = 0$). If $\alpha < \beta$ are both limit points of C, $i < \beta^+$ of $L(H(\beta))$ and $0 < n$, then we say that α is (i,n)-*Stable in* β if there is an $H \prec_{\Sigma_n} (L_i(H(\beta)), C \cap \beta)$ such that $H(\beta) \in H$ (if $i > 0$) and $H \cap H(\beta) = H(\alpha)$. In this case we let $H_n^{\beta,i}(\alpha)$ denote the $\subseteq$-smallest such H ([5]).

Note that α is $(0,n)$-Stable in β (β is $(0,n)$-Admissible) iff α is n-Stable in β (β is n-Admissible) via the earlier definition. We set:

$S^* = \{(\alpha, \beta, i, n) \mid \alpha \text{ is } (i,n)\text{-Stable in } \beta \text{ and } \beta \text{ is } (i,n)\text{-Admissible}\}$.

$\mathbb{S}^* = (L[S^*], S^*)$, the *Enriched Stable Core.*

Definition 8. Let (M, A) be an inner model of ZFC. Then a subclass Y of M is (M,A)-*constructible* if there exists a formula φ, parameter $p \in M$, club $D \subseteq \mathrm{Ord}$ and class $X \subseteq \mathrm{Ord}$ such that for α in D, $X \cap \alpha$ codes an ordinal i_α and $Y \cap H(\alpha)^M$ is definable over $(L_{i_\alpha}(H(\alpha)^M), A \cap H(\alpha)^M)$ via the formula φ with parameter p. If A is empty then we just say M-*constructible.*

Remark. If $(V, \mathcal{C})$ is a set model of Morse–Kelley and every linear order in $\mathcal{C}$ which $(V, \mathcal{C})$ thinks is a wellorder is really a wellorder, then the V-constructible classes in the sense of $(V, \mathcal{C})$ are exactly those subsets of V which belong to $L_\alpha(V)$ for some ordinal α which is the length of a wellorder in $\mathcal{C}$ ([6]). Moreover, if $\mathcal{C}_0$ consists of these classes then $(V, \mathcal{C}_0)$ is a model of Morse–Kelley which satisfies "every class is V-constructible".

Lemma 9 (Main Lemma). *Working in Gödel–Bernays, let V denote the sets and $\mathcal{C}$ denote the classes. Assume that every class is V-constructible. Then there is a (V, S^*)-definable class forcing $\mathbb{P}^*$ which adds a function from* Ord *to* 2 *such that, for $\mathbb{P}^*$-generic $F^* : \mathrm{Ord} \to 2$, $(V[F^*], \mathcal{C}[F^*])$ is a model of Gödel–Bernays minus Power (where $\mathcal{C}[F^*]$ consists of those classes which are definable in $(V[F^*], X, F^*)$ for some $X \in \mathcal{C}$), V is a definable inner*

([5]) To see that there is a smallest such H argue as follows. If $i = 0$ then of course $H(\alpha)$ itself is the smallest such H. Otherwise note that every element of $L_i(H(\beta))$ is Σ_1-definable in $L_i(H(\beta))$ from an ordinal less than i and parameters in $H(\beta) \cup \{H(\beta)\}$. Suppose that φ is a Σ_n formula with parameters from $H(\alpha) \cup \{H(\beta)\}$ and one free variable that has a solution in $(L_i(H(\beta)), C \cap \beta)$; we can choose a solution which is definable from parameters in $H(\alpha) \cup \{H(\beta)\}$ and an ordinal parameter $i_0 < i$ where the ordinal parameter i_0 has been minimised. But then i_0 belongs to any H which witnesses the (i,n)-Stability of α. Thus the $\subseteq$-smallest such H is the set of elements of $L_i(H(\beta))$ which arise in this way for some Σ_n formula φ with parameters from $H(\alpha) \cup \{H(\beta)\}$.

([6]) To see this, let Y be such a subset of V and suppose that Y is definable over $L_i(V)$ by the formula φ with parameters p and V; we may assume that p belongs to V by taking i to be least (if $i = 0$ then drop the parameter V). Choose D to be a club of (i,n)-Stables in $\mathrm{Ord}(V)$ such that p belongs to $H(\alpha)$ for α in D, and let X be a subset of $\mathrm{Ord}(V)$ that codes the ordinal i. Then φ, p, D and X witness the V-constructibility of Y. The converse follows by considering the structure $L_i(V)$ where X codes the ordinal i; by hypothesis X does indeed code an ordinal.

model of $(L[F^*], F^*)$ *and, for any* $\alpha < \beta$, $i < \beta^+$ *of* $L(H(\beta))$ *and* $0 < n$, *if* α *is* (i,n)*-Stable in* β *and* β *is* (i,n)*-Admissible then* α *is* (i,n)*-Stable in* β *relative to* F^* (*i.e. there is an* $H \prec_{\Sigma_n} (L_i(H(\beta)[F^* \restriction \beta]), C \cap \beta, F^* \restriction \beta)$ *such that* $H(\beta)[F^* \restriction \beta] \in H$ (*if* $i > 0$) *and* $H \cap H(\beta)[F^* \restriction \beta] = H(\alpha)[F^* \restriction \alpha]$).

REMARKS. (i) We do not expect $L[F^*]$ to satisfy the Power Set Axiom in general, and therefore in the above it is important to distinguish $H(\beta)[F^* \restriction \beta]$ from $H(\beta)^{L[F^*]}$; indeed the latter may fail to exist.

(ii) We do not require that if β is (i,n)-Admissible then this remains true relative to F^*, as we do not need it. However with some small modifications this could have been arranged as well.

Before proving the Main Lemma we describe its implications for the rigidity of HOD.

THEOREM 10. *Let* $\mathcal{C}^*$ *consist of the* $(L[S^*], S^*)$*-constructible classes, where* S^* *is the Enriched Stability predicate. Then:*

(1) $(L[S^*], \mathcal{C}^*)$ *has an outer model* $(L[F^*], \mathcal{C}^*[F^*])$ *of Gödel–Bernays minus Power which is generic over* $(L[S^*], \mathcal{C}^*)$ *for an* $\mathbb{S}^*$*-definable forcing which is* ∞*-cc* (*i.e. whose antichains in* $\mathcal{C}^*$ *are sets*) *such that* V *is a definable inner model of* $(L[F^*], F^*)$.

(2) $\mathbb{S}^*$ *is rigid in* $\mathcal{C}^*[F^*]$.

COROLLARY 11. *Assuming Morse–Kelley, any* V*-constructible club contains an* $(L[S^*], S^*)$*-constructible club and* $\mathbb{S}^* = (L[S^*], S^*)$ *is rigid for* V*-constructible embeddings.* (*It follows that also* (HOD, S^*) *is rigid for* V*-constructible embeddings.*)

Proof of Corollary 11 from Theorem 10. It suffices to show that assuming Morse–Kelley, any V-constructible class belongs to the $\mathcal{C}^*[F^*]$ of Theorem 10. Any such class belongs to a model A_V of KP + "every set is constructible from V" which is an end-extension of V, as Morse–Kelley is strong enough to produce such models (see for example [1]). Then A_V has an inner model $A_{L[S^*]} = (L[S^*])^{A_V}$ which is a model of KP + "every set is constructible from $L[S^*]$" which is an end-extension of $L[S^*]$. As V is a definable inner model of $(L[F^*], F^*)$ it follows that A_V is contained in $A_{L[S^*]}[F^*]$. But any class in $A_{L[S^*]}$ is $L[S^*]$-constructible and so the classes of $A_{L[S^*]}[F^*]$ belong to $\mathcal{C}^*[F^*]$. ∎

Proof of Theorem 10 from the Main Lemma. For conclusion (1) of the theorem of course we take F^* to be as in the Main Lemma and need to define an ∞-cc $\mathbb{S}^*$-definable forcing Q^* for which F^* is generic. In analogy to the case of the (unenriched) Stable Core we build the forcing Q^* out of quantifier-free infinitary sentences which belong to $L[S^*]$. Such sentences are obtained by closing the atomic sentences "$\dot{F}(\alpha) = 0$", "$\dot{F}(\alpha) = 1$" under infinitary conjunctions and disjunctions in $L[S^*]$. We let $\mathcal{L}^*$ denote the collection of such sentences which are consistent, i.e., which are true for some interpretation of $\dot{F}$ in a set-generic extension of $L[S^*]$; this notion of consistency is definable in $L[S^*]$.

Now we introduce a certain theory T^*, consisting of sentences of $\mathcal{L}^*$. For each $\alpha < \beta$, $i < \beta^+$ of $L(H(\beta))$ and $n > 0$ such that α is (i,n)-Stable in β and β is (i,n)-Admissible, and each set Φ of sentences of $\mathcal{L}^* \cap H(\beta)$ which is Σ_n-definable over $L_i(H(\beta)^{L[S^*]})$ with parameter p in $H(\alpha)^{L[S^*]}$, we insert the sentence

$$\bigwedge(\Phi \cap H(\alpha)) \to \bigwedge \Phi$$

into T^*. The forcing Q^* consists of all sentences φ of $\mathcal{L}^*$ which are consistent with T^* (i.e. $\bigwedge(T_0^* \cup \{\varphi\})$ is consistent for each $T_0^* \subseteq T^*$, $T_0^* \in L[S^*]$). We order Q^* by $\varphi \le \psi$ iff $\varphi \wedge {\sim}\psi$ is not consistent with T^*.

The sentences in T^* are all true when $\dot{F}$ is interpreted as F^*, thanks to the fact that F^* preserves instances of (i,n)-Stability.

FACT 1. *The forcing Q^* is ∞-cc in $\mathcal{C}^*$.*

Proof. Let A be a maximal antichain on Q^* which is $L[S^*]$-constructible and choose a wellorder $<$, club D, parameter p and φ that witness the $L[S^*]$-constructibility of A. Let φ be Σ_n. Let α be the least element of D; we claim that $A = A \cap H(\alpha)$ and therefore A is a set in $L[S^*]$. Indeed, for any β in D, the axioms of T yield $\bigvee(A \cap H(\beta)) \to \bigvee(A \cap H(\alpha))$ by virtue of the (i,n)-Stability of α in β where $i = \mathrm{ot}(<\restriction\beta)$. As A is an antichain, $A \cap H(\alpha)$ must equal all of $A \cap H(\beta)$ for each β in D, and as D is unbounded, $A \cap H(\alpha)$ equals all of A. ∎

Let G^* consist of all sentences of $\mathcal{L}^*$ which are true when $\dot{F}$ is interpreted as F^*. Clearly G^* intersects each maximal antichain A of Q^* which is a set in $L[S^*]$, as otherwise $\bigwedge_{\varphi \in A} {\sim}\varphi$ would be a sentence consistent with T (and therefore in $\mathcal{L}^*$) violating the maximality of A. But by Fact 1, all antichains of Q^* in $\mathcal{C}^*$ are sets in $L[S^*]$ and so G^* is fully Q^*-generic over $(L[S^*], \mathcal{C}^*)$. This establishes conclusion (1) of the theorem.

For (2) we give two proofs. The first is simpler, but appears to need Morse–Kelley in $(V, \mathcal{C})$ as the background theory. (It is sufficient for establishing Corollary 11, but not for the more general results mentioned in the Abstract.)

First proof. Suppose that $j : \mathbb{S}^* \to \mathbb{S}^*$ is not the identity and j belongs to $\mathcal{C}^*[F^*]$. Assuming Morse–Kelley in $(V, \mathcal{C})$ (and therefore Morse–Kelley minus Power in $(L[F^*], \mathcal{C}^*[F^*])$), we show that j can be extended to $j^* : (L[S^*], \mathcal{C}^*) \to (L[S^*], \mathcal{C}^*)$. Indeed, for each ordinal α, each class $X \in \mathcal{C}^*$ which codes a sequence of classes $(X_i \mid i \in \mathrm{Ord})$ and each $i \in \mathrm{Ord}$ let $H(\alpha, X, i)$ consist of all elements of the structure $(L[S^*], \{X_i \mid i \in \mathrm{Ord}\})$ which are definable with parameters from $\alpha \cup \{i\}$. We write $(\beta, Y, j) >$

(α, X, i) iff $\beta > \alpha$, $X = Y_k$ for some $k < \beta$ and $i < \beta$; this implies that $H(\beta, Y, j)$ contains $H(\alpha, X, i)$ as a substructure. The structures $H(\alpha, X, i)$ ordered by $<$ form a direct system which is isomorphic to a direct system whose elements and maps belong to $L[S^*]$. We can apply j to the elements of maps of this system Π to obtain a system $j[\Pi]$ whose limit is isomorphic to $(L[S^*], \mathcal{C}^*)$, using the fact that $\mathcal{C}^*$ consists only of the $\mathbb{S}^*$-constructible classes. This yields an elementary embedding $j^* : (L[S^*], \mathcal{C}^*) \to (L[S^*], \mathcal{C}^*)$ as desired. Note that j^* can be simply defined by setting $j^*(X) = \bigcup_{\alpha \in \mathrm{Ord}} j(X \cap L_\alpha[S^*])$. However to establish the elementarity of this j^* we appear to need the argument with direct limit systems given.

But now we can proceed as in the first proof of Corollary 6: The embedding j^* is definable over $(L[F^*], \mathcal{C}^*[F^*])$ and therefore generic over $(L[S^*], \mathcal{C}^*)$ for an ∞-cc definable forcing. The least ordinal forced by some condition in this forcing to be the critical point of such an embedding is $(L[S^*], \mathcal{C}^*)$-definable and therefore cannot be moved by such an embedding, a contradiction.

Second proof. We only assume that $(V, \mathcal{C})$ models Gödel–Bernays, and need two facts.

Fact 2. There is an $(L[S^*], S^*)$-definable $\diamondsuit$-sequence $(S_\alpha \mid \alpha \in \mathrm{Ord})$ for $(L[S^*], \mathcal{C}^*)$ which concentrates on strong limit cardinals of cofinality ω of $L[S^*]$; that is, if X belongs to $\mathcal{C}^*$ and D is a club in $\mathcal{C}^*$ then there is a strong limit cardinal α of cofinality ω of $L[S^*]$ such that $X \cap \alpha = S_\alpha$.

Proof. Let S_α be empty if α is not a limit point of C which in addition is a strong limit cardinal of cofinality ω of $L[S^*]$. Otherwise, assuming that S_β is defined for $\beta < \alpha$ we take (S_α, C_α) to be the least pair in $L(H(\alpha)^{L[S^*]})$ such that C_α is closed unbounded in α and $S_\alpha \cap \bar{\alpha} \neq S_{\bar{\alpha}}$ for $\bar{\alpha}$ in C_α, if it exists, $(\emptyset, \emptyset)$ otherwise. (Note that even though α has cofinality ω, we can still talk about closed unbounded subsets of α, which indeed may appear at a level of $L(H(\alpha)^{L[S^*]})$ before it is recognised that α is singular.) Suppose that the resulting sequence is not the desired $\diamondsuit$-sequence and let (S, D) in $\mathcal{C}^*$ be a counterexample, i.e., D is a club and for limit points α of C which are strong limit cardinals of cofinality ω of $L[S^*]$ in D, $S \cap \alpha \neq S_\alpha$. Then for each α in D (which is a limit point of C and a strong limit cardinal of cofinality ω of $L[S^*]$), the pair (S_α, C_α) was chosen as the least pair such that $S_\alpha \cap \bar{\alpha} \neq S_{\bar{\alpha}}$ for $\bar{\alpha}$ in C_α. But this choice of S_α is Σ_1-definable in $L_{\mathrm{ot}(<\restriction\alpha)}(H(\alpha)^{L[S^*]})$ for a club E of α's, where $<$, E belong to $\mathcal{C}^*$ and witness the $L[S^*]$-constructibility of (S, D); moreover, E can be chosen so that there is a Σ_1-elementary embedding of $L_{\mathrm{ot}(<\restriction\alpha)}(H(\alpha)^{L[S^*]})$ into $L_{\mathrm{ot}(<\restriction\beta)}(H(\beta)^{L[S^*]})$ for $\alpha < \beta$ in E. It follows that $S_\beta \cap \alpha = S_\alpha$, $C_\beta \cap \alpha = C_\alpha$ for $\alpha < \beta$ in E. This is a contradiction as we can choose $\alpha < \beta$ in $E \cap D$ to be limit points of C which are strong limit cardinals of cofinality ω of $L[S^*]$, yielding $S_\beta \cap \alpha = S_\alpha$ with α in C_β. ∎

FACT 3. Any club in $\mathcal{C}^*[F^*]$ contains a club in $\mathcal{C}^*$.

Proof. This is because, by Fact 1, $(L[F^*], \mathcal{C}^*[F^*])$ is an ∞-cc generic extension of $(L[S^*], \mathcal{C}^*)$. ∎

Now for the rigidity of $\mathbb{S}^*$ in $\mathcal{C}^*[F^*]$ we argue as before: Using Fact 2 we can obtain an $\mathbb{S}^*$-definable partition $(T_\alpha \mid \alpha \in \mathrm{Ord})$ of the ordinals of cofinality ω into pieces which are $\mathcal{C}^*$-stationary, i.e., which intersect any club in $\mathcal{C}^*$. By Fact 3 any club in $\mathcal{C}^*[F^*]$ contains a club in $\mathcal{C}^*$. But now there can be no nontrivial elementary embedding $j : \mathbb{S}^* \to \mathbb{S}^*$ in $\mathcal{C}^*[F^*]$: otherwise we can choose α in $j((T_\alpha \mid \alpha \in \mathrm{Ord}))_\kappa$ to be a fixed point of j and derive the contradiction that α belongs to both $j((T_\alpha \mid \alpha \in \mathrm{Ord}))_\kappa$ and $j((T_\alpha \mid \alpha \in \mathrm{Ord}))_\gamma$ for some $\gamma < \kappa$. This completes the second proof of Theorem 10(2). ∎

Proof of the Main Lemma. The desired forcing $\mathbb{P}^*$ is the final stage $\mathbb{Q}^*_\infty$ of a *finite support* iteration $(\mathbb{P}^*_\beta, \mathbb{Q}^*_\beta \mid \beta \in C \cup \{\infty\})$. The βth stage $\mathbb{Q}^*_\beta$ of the iteration will add a function $p^* : \beta \to 2$. If $\beta = \omega$ is the minimum of C then $\mathbb{Q}^*_\beta$ is the atomic forcing whose conditions are functions $p^* : \omega \to 2$. If β is a successor point of C and β_0 is its C-predecessor then $\mathbb{Q}^*_\beta$ is an atomic forcing, whose conditions consist of all $p^* : \beta \to 2$ in $V[G^*_{\beta_0}, G^*(\beta_0)]$ such that $p^* \restriction \beta_0$ is $\mathbb{Q}^*_{\beta_0}$-generic over $V[G^*_{\beta_0}]$ (where G^*_α, $G^*(\alpha)$ denote the generics for $\mathbb{P}^*_\alpha$, $\mathbb{Q}^*_\alpha$ respectively for each α in C); we also require that $p^* \restriction [\beta_0, \beta)$ belong to V, $p^*(\beta_0) = 1$ and $p^*(2\gamma) = 0$ for all γ in (β_0, β). (These latter requirements ensure that both V and C are definable over $(L[F^*], F^*)$ when $F^* : \mathrm{Ord} \to 2$ is $\mathbb{P}^*$-generic.)

Suppose that β is a limit point of C. Let $\mathbb{Q}^{*,0}_\beta$ denote the set (or class if $\beta = \infty$) of all $p^* : \alpha \to 2$ in $V[G^*_\alpha, G^*(\alpha)]$ where $\alpha \in C \cap \beta$ and $p^* \restriction \alpha$ is $\mathbb{Q}^*_\alpha$-generic over $V[G^*_\alpha]$; $\mathbb{Q}^{*,0}_\beta$ is ordered by extension. If β is regular in $L(H(\beta))$ or $\beta = \infty$ then $\mathbb{Q}^*_\beta$ is equal to $\mathbb{Q}^{*,0}_\beta$. Otherwise, proceed as follows. We say that $p^* : \beta \to 2$ is (i,n)-*generic for* $\mathbb{Q}^{*,0}_\beta$ if $G^*(p^*) = \{p^* \restriction \alpha \mid \alpha \in C \cap \beta\}$ meets every dense subset of $\mathbb{Q}^{*,0}_\beta$ of the form $\{q^* \in \mathbb{Q}^{*,0}_\beta \mid q^* \Vdash \varphi \text{ or } q^* \Vdash \sim\varphi\}$, where φ is a $\Pi_n(L_i(H(\beta)), C \cap \beta, \dot{f})$ sentence with parameters from $H(\beta) \cup \{H(\beta)\}$ (just parameters from $H(\beta)$ if $i = 0$; here $\dot{f}$ denotes the generic function with domain β). Then we take $\mathbb{Q}^*_\beta$ to be the atomic forcing whose conditions are functions $p^* : \beta \to 2$ in $V[G^*_\beta]$ which are (i,n)-generic for $\mathbb{Q}^{*,0}_\beta$ for the (fewer than β^+ of $L(H(\beta))$-many) (i,n) such that β is (i,n)-Admissible.

For notational convenience, we define $\mathbb{Q}^{*,0}_\beta$ to be $\mathbb{Q}^*_{\beta_0}$ when β is a successor point of C and β_0 is its C-predecessor.

Lemma 12. *Suppose that β belongs to C and β is either a successor point of C or not regular in $L(H(\beta))$. Then, in $V[G^*_\beta]$, each p^* in $\mathbb{Q}^{*,0}_\beta$ has an extension in $\mathbb{Q}^*_\beta$.*

Proof. We use induction on β. Suppose that β is a successor point of C and let β_0 be its C-predecessor. If $\beta_0 = \omega$ is the minimum of C then it is easy to extend any element of $\mathbb{Q}^*_{\beta_0}$ to an element of $\mathbb{Q}^*_\beta$. If β_0 is a successor point of C or not regular in $L(H(\beta_0))$ then by induction, in $V[G^*_{\beta_0}]$, each p^* in $\mathbb{Q}^{*,0}_{\beta_0}$ has an extension p^{**} in $\mathbb{Q}^*_{\beta_0}$; it is then easy to extend p^{**} further to an element of $\mathbb{Q}^*_\beta$. If β_0 is a limit point of C and is regular in $L(H(\beta_0))$ then by induction any p^* in $\mathbb{Q}^{*,0}_{\beta_0}$ has extensions in $\mathbb{Q}^*_\gamma$ for arbitrarily large $\gamma \in C \cap \beta_0$; it follows that any $\mathbb{Q}^*_{\beta_0}$-generic p^{**} has domain β_0 and it then follows that each p^* in $\mathbb{Q}^{*,0}_{\beta_0}$ can be extended to some $\mathbb{Q}^*_{\beta_0}$-generic p^{**} in $V[G^*_\beta]$ (the forcing $\mathbb{Q}^*_{\beta_0}$ is homogeneous). It is then easy to extend p^{**} further to an element of $\mathbb{Q}^*_\beta$ in $V[G^*_\beta]$.

Suppose that β is a limit point of C and is not regular in $L(H(\beta))$. Let $(i, n+1)$ be least so that β is not $(i, n+1)$-Admissible ([7]). First suppose that $n = 0$. If $i = 0$ then β is not 1-Admissible and there is a closed unbounded subset D of $C \cap \beta$ of ordertype less than β whose successor points γ are not regular in $L(H(\gamma))$ and whose intersection with each of its limit points $\gamma < \beta$ is Δ_1-definable over $(H(\gamma), C \cap \gamma)$. Given $\alpha \in C \cap \beta$ and a p^* in $\mathbb{Q}^{*,0}_\beta$ that we want to extend into $\mathbb{Q}^*_\beta$, we can assume that both α and the ordertype of D are less than the minimum of D. Now enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p^* to $q^*_0 \subseteq q^*_1 \subseteq \cdots$ with q^*_j in $\mathbb{Q}^*_{\beta_j}$, taking unions at limits. Note that for limit j, q^*_j is indeed a condition because β_j is not 1-Admissible. The union of the q_j's is the desired extension of p^* in $\mathbb{Q}^*_\beta$. If $i = i_0+1$ is a successor ordinal then we instead choose D to be a closed unbounded subset of $C \cap \beta$ of ordertype less than β whose successor points γ are not regular in $L(H(\gamma))$ and such that for limit points $\gamma < \beta$ of D, $D \cap \gamma$ is Δ_1-definable over the transitive collapse of the hull $H^{\beta,i_0}_\omega(\gamma)$ (= the $\subseteq$-least Σ_ω-elementary submodel H of $(L_{i_0}(H(\beta)), C \cap \beta)$ containing $H(\beta)$ as an element (if $i_0 > 0$) such that $H \cap H(\beta) = H(\gamma)$). Again we make successive extensions of p^* to $q^*_0 \subseteq q^*_1 \subseteq \cdots$ with q^*_j in $\mathbb{Q}^*_{\beta_j}$, taking unions at limits, where the β_j's increasingly enumerate D. We get a condition at limit stages using the fact that γ is not Σ_1-regular over the transitive collapse of $H^{\beta,i_0}_\omega(\gamma)$ when it is a limit point of D (and using reflection to infer that the associated limit q^*_j is indeed sufficiently generic for the forcing $\mathbb{Q}^{*,0}_\gamma$).

Now suppose that $n > 0$.

([7]) Note that if i is least so that β is not $\Sigma_\omega(L_i(H(\beta)))$-regular then β is not $(i, n+1)$-Admissible for some n.

If β is a limit of α which are (i,n)-Stable in β then proceed as in the previous paragraph: Choose a closed unbounded subset D of $C\cap\beta$ of ordertype less than β consisting of α which are (i,n)-Stable in β, whose successor points γ are not regular in $L(H(\gamma))$ and whose intersection with each of its limit points $\gamma<\beta$ is Δ_{n+1}-definable over the transitive collapse of $H_n^{(i,\beta)}(\gamma)$. Assume that the ordertype of D as well as the domain of the given $p^*\in\mathbb{Q}_\beta^{*,0}$ that we wish to extend are less than the minimum of D, enumerate D as $\beta_0<\beta_1<\cdots$ and, using the induction hypothesis, successively extend p to $q_0\subseteq q_1\subseteq\cdots$ with q_j in $\mathbb{Q}^*_{\beta_j}$, taking unions at limits. For limit j, q_j is indeed a condition because β_j is not $(\bar{i},n+1)$-Admissible, where $\bar{i}$ is the height of the transitive collapse of $H_n^{(i,\beta)}(\beta_j)$, and as it is a limit of ordinals which are (i,n)-Stable in β, q_j is $(\bar{i},n)$-generic for $\mathbb{Q}^{*,0}_{\beta_j}$. The union of the q_j's is the desired extension of p^* in $\mathbb{Q}^*_\beta$.

If β is not a limit of α which are (i,n)-Stable in β then β must have cofinality ω (else by (i,n)-Admissibility, we could find cofinally many (i,n)-Stables in β, for $i>0$ using the fact that the subsets of $H(\beta)$ which are Σ_n-definable over $(L_i(H(\beta)),C\cap\beta)$ with parameters from $H(\beta)\cup\{H(\beta)\}$ are those which are Σ_1-definable over $(H(\beta),T_{n-1})$ where T_{n-1} is the Σ_{n-1} theory of $(L_i(H(\beta)),C\cap\beta)$ with parameters from $H(\beta)\cup\{H(\beta)\}$). It suffices to show that any condition p^* in $\mathbb{Q}_\beta^{*,0}$ can be extended to decide (i.e. force or force the negation of) each of fewer than β-many $\Pi_n(L_i(H(\beta)),C\cap\beta)$ sentences with parameters from $H(\beta)\cup\{H(\beta)\}$ (just $H(\beta)$ if $i=0$). (Given this, we can extend p^* in ω steps to a condition in $\mathbb{Q}^*_\beta$ which is (i,n)-generic for $\mathbb{P}^*_\beta$.) To show this, let $(\varphi_j\mid j<\delta)\in H(\beta)$ enumerate the given collection of $\Pi_n(L_i(H(\beta)),C\cap\beta)$ sentences (by explicitly listing the sentences with their parameters from $H(\beta)$, treating the parameter $H(\beta)$ as implicit, if $i>0$), and if $n>1$, let D consist of all γ which are limits of $(i,n-1)$-Stables in β and large enough so that $H(\gamma)$ contains p^* and $H^{i,\beta}_{n-1}(\gamma)$ contains this enumeration. (If $n=1$ then let D consist of all γ which are limit points of C and large enough so that $H(\gamma)$ contains p^* and this enumeration.) Now extend p^* successively to elements q_j of $\mathbb{Q}^*_{\gamma_j}$, where $\gamma_{j+1}\geq\gamma_j$ is the least element γ of D so that γ is not regular in $L(H(\gamma))$ and either q_j forces φ_j or q_{j+1} forces $\psi_j=$ the negation of φ_j (with corresponding witness to the Σ_n sentence ψ_j), taking unions at limits. For limit j, q_j is a condition because γ_j is not $(\bar{i},n)$-Admissible but (in case $n>1$) is a limit of $(\bar{i},n-1)$-Stables, where $\bar{i}$ is the height of the transitive collapse of $H_{n-1}^{(i,\beta)}(\gamma_j)$. (The failure of γ_j to be $(\bar{i},n)$-Admissible uses the fact that the set of $j_0<j$ such that q_{j_0+1} forces the negation of φ_{j_0} can be treated as a parameter in $H(\gamma_j)$.) As β is (i,n)-Admissible, this construction results in a sequence of q_j's of length δ, whose union is the desired extension of p^* deciding all of the given $\Pi_n(L_i(H(\beta)),C\cap\beta)$ sentences. ∎

Lemma 13. *Suppose that* G^* *is* $\mathbb{Q}^*_\infty$*-generic where* $\mathbb{Q}^*_\infty$ *is the class of* $p^* : \alpha \to 2$ *in* $V[G^*_\infty]$ *such that* α *belongs to* C *and* p^* *is* $\mathbb{Q}^*_\alpha$*-generic. Let* $F^* : \mathrm{Ord} \to 2$ *be the union of the functions in* G^**. Then* V *is a definable inner model of* $L[F^*]$ *and, for any* $\alpha < \beta$, $i < \beta^+$ *of* $L(H(\beta))$ *and* $0 < n < \omega$, *if* α *is* (i, n)*-Stable in* β *and* β *is* (i, n)*-Admissible then* α *is* (i, n)*-Stable in* β *relative to* F^*.

Proof. It is easy to define V from F^*, as from F^* we can first identify the elements of C and then V consists of those sets coded by F^* restricted to some adjacent interval of C. Suppose that α is (i, n)-Stable in β and β is (i, n)-Admissible. Then by the definition of $\mathbb{Q}^*_\infty$, $F^*\restriction\beta$ is (i, n)-generic for $\mathbb{Q}^{*,0}_\beta$ and $F^*\restriction\alpha$ is $(\bar{i}, n)$-generic for $\mathbb{Q}^{*,0}_\alpha$ where $H^{(i,\beta)}_n(\alpha)$ has transitive collapse of height $\bar{i}$, as α is $(\bar{i}, n)$-Admissible. But as the forcing relation for Π_n formulas is Π_n-definable, this implies that α is (i, n)-Stable in β relative to F^*, as desired. ■

Now notice that since we iterate with finite support, the forcing $\mathbb{P}^*_\infty$ is ∞-cc, i.e., all antichains for this forcing which belong to $\mathcal{C}$ are sets in V. It follows that Gödel–Bernays minus Power is preserved. This completes the proof of the Main Lemma and therefore of Theorem 10. ■

Open questions. Can one prove in Morse–Kelley (or even in Gödel–Bernays) that HOD is relatively rigid for arbitrary class embeddings? Is HOD rigid (not just relatively rigid) for V-constructible classes?

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