Descriptive Set Theory on Generalised Baire Space

Joint work with Khomskii-Kulikov (first part) and with Hyttinen-Kulikov (second part)

We assume $\kappa = \kappa^{<\kappa}$.

κ-*Baire space* $= \kappa^\kappa$ consists of all $f : \kappa \to \kappa$, with basic open sets given by

$$\{f : \kappa \to \kappa \mid s \subseteq f\}$$

where $s \in \kappa^{<\kappa}$.

Nowhere dense = Closure has no interior
Meager = union of κ-many nowhere dense sets

Baire measurable = differs from an open set by a meager set

The Baire Category theorem holds (the intersection of κ-many open dense sets is dense)

Regularity Properties

Baire measurability is just one example of a regularity property.

A forcing $\mathcal{P}$ is κ-*treelike* iff it is a κ-closed suborder of the set of subtrees of $\kappa^{<\kappa}$, ordered by inclusion.

Some examples of κ-treelike forcings:

κ-*Cohen* $\mathbb{C}_\kappa$. These are subtrees of $2^{<\kappa}$ consisting of a stem and all nodes above it.

κ-*Sacks* $\mathbb{S}_\kappa$. These are κ-closed subtrees of $2^{<\kappa}$ with the property that every node has a splitting extension and the limit of splitting nodes is a splitting node.

κ-*Silver* $\mathbb{V}_\kappa$, for inaccessible κ. These are κ-Sacks trees T which are *uniform*, i.e. if s, t are elements of T of the same length then $s * i$ is in T iff $t * i$ is in T for $i = 0, 1$.

Regularity Properties

κ-*Miller* $\mathbb{M}_\kappa$. These are κ-closed subtrees of the tree $\kappa^{<\kappa}_{\uparrow}$ of *increasing* sequences in $\kappa^{<\kappa}$ with the property that every node can be extended to a club-splitting node and the limit of club-splitting nodes is club-splitting. We also require *continuous club-splitting*, which means that if s is a limit of club-spltting nodes then the club witnessing club-splitting for s is the intersection of the clubs witnessing club-splitting for the club-splitting proper initial segments of s.

κ-*Laver* $\mathbb{L}_\kappa$. These are κ-Miller trees with the property that every node beyond some fixed node (the stem) is club-splitting.

κ-*Mathias* $\mathbb{R}_\kappa$. Conditions are pairs (s, C) where s is a bounded subset of κ and C is a club in κ. $(t, D) \leq (s, C)$ iff t end-extends s, $D \subseteq C$ and $t \setminus s \subseteq C$. This is equivalent to a κ-treelike forcing.

Regularity Properties

The 6 examples above fall into two groups:

$\mathbb{C}_\kappa$, $\mathbb{L}_\kappa$ and $\mathbb{R}_\kappa$ are *topological*: The $[T]$ for $T \in \mathcal{P}$ form the base for a topology (either $[S] \cap [T]$ is empty or contains some $[U]$). They are κ^+-cc.

$\mathbb{S}_\kappa$, $\mathbb{M}_\kappa$ and $\mathbb{V}_\kappa$ are not κ^+-cc but they satisfy a form of *fusion* (called *Axiom A**), sufficient to show that κ^+ is preserved.

Remark. There is no obvious κ-analogue of Solovay forcing (random real forcing). However:

Theorem

(SDF-Laguzzi) If $V = L$ and κ is inaccessible then there is a Δ^1_1 κ-treelike forcing $\mathbb{B}_\kappa$ which is κ^+-cc and κ^κ-bounding.

Regularity Properties

To define "$\mathcal{P}$-measurability" for κ-treelike forcings $\mathcal{P}$ we proceed as follows.

A set A is:

Strictly $\mathcal{P}$-null if every tree $T \in \mathcal{P}$ has a subtree in $\mathcal{P}$, none of whose κ-branches belongs to A.

$\mathcal{P}$-null (or *$\mathcal{P}$-meager*) if it is the union of κ-many strictly $\mathcal{P}$-null sets.

$\mathcal{P}$-measurable (or *$\mathcal{P}$-regular*) if any tree $T \in \mathcal{P}$ has a subtree $S \in \mathcal{P}$ such that either all κ-branches through S, with a $\mathcal{P}$-null set of exceptions, belong to A or all κ-branches through S, with a $\mathcal{P}$-null set of exceptions, belong to the complement of A.

Regularity Properties

Proposition

(a) If $\mathcal{P}$ is topological then:
(a1) A set is $\mathcal{P}$-measurable iff it differs from a $\mathcal{P}$-open set by a $\mathcal{P}$-null set. (So $\mathbb{C}_\kappa$-measurable is the same as Baire-measurable.)
(a2) Not every $\mathcal{P}$-null set is strictly $\mathcal{P}$-null.
(a3) Borel sets are $\mathcal{P}$-measurable.
(b) If $\mathcal{P}$ satisfies fusion (Axiom A) then:*
(b1) Every $\mathcal{P}$-null set is strictly $\mathcal{P}$-null.
(b2) Borel sets are $\mathcal{P}$-measurable.

Question. As in the case $\kappa = \omega$, are all Σ^1_1 sets $\mathcal{P}$-measurable?

Answer: NO!

Regularity Properties

Fact. The club filter $= \{f : \kappa \to 2 \mid f(i) = 1$ for club-many $i < \kappa\}$ is not κ-Sacks ($\mathbb{S}_\kappa$) measurable.

Proof. Otherwise there is a κ-Sacks tree T such that either for all $f \in [T]$, $f(i) = 1$ for club-many $i < \kappa$ or for all $f \in [T]$, $f(i) = 0$ for stationary-many $i < \kappa$.

But we can easily build f_0, f_1 in $[T]$ such that whenever $f_0|i$ splits in T, $f(i) = 0$ and whenever $f_1|i$ splits in T, $f(i) = 1$.
And the set of i where $f_0|i$ splits forms a club (same for f_1).

So $[T]$ has an element f_0 which is not in the club filter and an element f_1 which is. $\square$

Regularity Properties

Now we can apply the following result to conclude that Σ^1_1 sets need not be $\mathcal{P}$-measurable for any of our 6 examples. For a pointclass Γ, let $\Gamma(\mathcal{P})$ denote that sets in Γ are $\mathcal{P}$-measurable.

Theorem

(a) $\Gamma(\mathbb{C}_\kappa) \to \Gamma(\mathbb{V}_\kappa) \to \Gamma(\mathbb{S}_\kappa)$.
(b) $\Gamma(\mathbb{C}_\kappa) \to \Gamma(\mathbb{M}_\kappa) \to \Gamma(\mathbb{S}_\kappa)$.
(c) $\Gamma(\mathbb{R}_\kappa) \to \Gamma(\mathbb{M}_\kappa)$.
(d) $\Gamma(\mathbb{L}_\kappa) \to \Gamma(\mathbb{M}_\kappa)$.

In particular $\Gamma(\mathbb{S}_\kappa)$ is the weakest of them all, so as it fails for $\Gamma = \Sigma^1_1$ so do all the others.

Question. What about Δ^1_1 ($\neq$ Borel for $\kappa > \omega$)?

Regularity Properties

Theorem

It is consistent to have $\Delta^1_1(\mathcal{P})$ *for* $\mathcal{P} = \mathbb{C}_\kappa$, $\mathbb{L}_\kappa$ *and* $\mathbb{R}_\kappa$ *simultaneously.*

This is proved by interleaving iterations with $< \kappa$-support of these three forcings for κ^+ steps.

Note that in the above model we also have $\Delta^1_1(\mathcal{P})$ for $\mathcal{P} = \mathbb{M}_\kappa$, $\mathbb{V}_\kappa$ and $\mathbb{S}_\kappa$, by the previous slide.

Question. But can we separate $\Delta^1_1(\mathcal{P})$ for different $\mathcal{P}$?

This looks hard. But we have one result about it:

Regularity Properties

Theorem

There is a model where κ is inaccessible and $\Delta^1_1(\mathbb{V}_\kappa)$ holds but $\Delta^1_1(\mathbb{M}_\kappa)$ fails.

This is proved by iterating $\mathbb{V}_\kappa$ for κ^+ steps over L, where κ is inaccessible; $\Delta^1_1(\mathbb{V}_\kappa)$ holds in the resulting model.
The main lemma is that $\Delta^1_1(\mathbb{M}_\kappa)$ yields functions from κ to κ that are unbounded over $L[f]$, for any given $f : \kappa \to \kappa$.
As the iteration is κ^κ-bounding and therefore does not add functions which are unbounded over the ground model, we conclude that $\Delta^1_1(\mathbb{M}_\kappa)$ fails.

It follows from our earlier implications between regularity properties that in the above model, $\Delta^1_1(\mathbb{C}_\kappa)$, $\Delta^1_1(\mathbb{R}_\kappa)$ and $\Delta^1_1(\mathbb{L}_\kappa)$ all fail, but $\Delta^1_1(\mathbb{S}_\kappa)$ holds.

Regularity Properties

The main difficulty with separating Δ^1_1 regularity properties is the lack of "Solovay-type characterisations".
In the classical setting we have:

(Solovay) Σ^1_2 sets are Baire-measurable iff for every real x there is a comeager set of reals Cohen over $L[x]$.
(Shelah) Δ^1_2 sets are Baire-measruable iff for every real x there is a Cohen real over $L[x]$.

In fact, Shelah's result provably fails for uncountable κ:

Theorem

(SDF-Wu-Zdomskyy) Suppose that κ is regular and uncountable in L. Then in a cofinality-preserving forcing extension, for every $x \subseteq \kappa$ there is a κ-Cohen over $L[x]$ but the CUB filter on κ is Δ^1_1. In particular not all Δ^1_1 sets are Baire-measurable.

Borel Reducibility

If E and F are equivalence relations on κ^κ then we say that E *is Borel reducible to* F, written $E \leq_B F$, if there is a Borel function f such that for all x, y: $E(x, y)$ iff $F(f(x), f(y))$. The relation $\leq_B$ is reflexive and transtive and we write $\equiv_B$ for the equivalence relation it induces.

Borel Reducibility: Dichotomies

In the classical setting one has two important Dichotomies:

Silver Dichotomy. Suppose that E is a Borel equivalence relation on ω^ω with uncountably many classes. Then equality is Borel (even continuously) reducible to E.

Harrington-Kechris-Louveau Dichotomy. Suppose that E is a Borel equivalence relation. Then either E is Borel reducible to equality or E_0 is Borel reducible to E, where E_0 is the equivalence relation of equality mod finite.

In generalised Baire space, the Silver Dichotomy fails in L but consistently holds (after collapsing a Silver indiscernible to become ω_2), and the Harrington-Kechris-Louveau Dichotomy simply fails.

Borel Reducibility: Small Equivalence Relations

Theorem

If E is the orbit equivalence relation of a Borel action of a group of size at most κ then E is Borel reducible to E_0.

Proof. The key observation is this: Let F_κ denote the free group on κ generators. Then F_α has cardinality less than κ for $\alpha < \kappa$ (this fails when κ equals ω). Using this one shows that the shift action of F_κ (sending (g, X) in $G \times \mathcal{P}(F_\kappa)$ to $\{g \cdot x \mid x \in X\}$) reduces to E_0: Map $X \subseteq F_\kappa$ to the sequence $f(X) = (<_\alpha$-least element of $\{g_\alpha \cdot (X \cap F_\alpha) \mid g_\alpha \in F_\alpha\} \mid \alpha < \kappa)$. If X, Y are equivalent under shift then it is easy to check $f(X) E_0 f(Y)$; the converse uses Fodor's theorem. $\square$

Borel Reducibility: Small Equivalence Relations

Theorem

Assume $V = L$. Then there is a smooth Borel equivalence relation with classes of size 2 which is not induced by a Borel action of a small group.

Proof. Let X be the Borel set of Master Codes for initial segments of L of size κ and $\sim X$ its complement. Define a bijection $f : \sim X \to X$ with Borel graph and define $E(x, y)$ iff $y = f(x)$ or $x = f(y)$. Then E is smooth. If it were induced by a Borel action of a group of size at most κ then f would be Borel on a non-meager set, which is impossible. $\square$

Borel Reducibility: E_1

Theorem

E_1 is Borel reducible to E_0.

Proof idea: For limit $\alpha < \kappa$, define E_1^α to be the equivalence relation on $(2^\alpha)^\alpha$ approximating E_1 defined by $(x_i)_{i<\alpha} E_1^\alpha (y_i)_{i<\alpha}$ iff for some $\beta < \alpha$, $x_i = y_i$ for all $i > \beta$.
Now define $F((x_i)_{i<\kappa})(\alpha)$ to be 0 if α is not a limit and otherwise to be a code for the E_1^α-equivalence class of $(x_i \restriction \alpha)_{i<\alpha}$.
Clearly if $(x_i)_{i<\kappa} E_1 (y_i)_{i<\kappa}$ then $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are E_0-equivalent.
Conversely, if $(x_i)_{i<\kappa}$ and $(y_i)_{i<\kappa}$ are not E_1 equivalent then for club-many $\alpha^* < \kappa$, $(x_i \restriction \alpha^*)_{i<\alpha^*}$ and $(y_i \restriction \alpha^*)_{i<\alpha^*}$ are not $E_1^{\alpha^*}$-equivalent; it follows that $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are not E_0-equivalent. $\square$

Borel Reducibility: Isomorphism Relations

Theorem

(a) Each Borel isomorphism relation is Borel reducible to the α-th jump of equality for some $\alpha < \kappa^+$.
(b) For each $\alpha < \kappa^+$, the α-th jump of equality is Borel reducible to equality on κ^κ modulo a μ-nonstationary set, for any regular $\mu < \kappa$.
(c) A first-order theory is classifiable and shallow iff the isomorphism relation on its models of size κ is Borel.
(d) (For a suitable successor κ) A first-order theory is unclassifiable iff equality on 2^κ modulo a μ-nonstationary set is Borel reducible to the isomorphism relation on its models of size κ for some regular $\mu < \kappa$.

Is equality on κ^κ modulo a μ-nonstationary set Borel reducible to equality on 2^κ modulo a μ-nonstationary set?
If so we have:

Borel Reducibility: Isomorphism Relations

If T_0 is classifiable and shallow and T_1 is unclassifiable then isomorphism on the models of T_0 of size κ is Borel reducible to isomorphism on the models of T_1 of size κ (for example when κ is the successor of an uncountable regular and GCH holds).