7 Aspects of Pure Set Theory

Dresden Lecture, 23. September, 2000

Set theory entered the modern era through the work of Gödel and Cohen, which provided set-theorists with the necessary tools to deal with the undecidability of a large number of mathematical questions. Through these methods, together with their generalisation into the context of large cardinals, set-theorists have had great success in determining the axiomatic strength of a wide range of problems, not only within set theory but also within other areas of mathematics.

This talk is primarily concerned with *pure set theory*, focusing on the techniques that are needed in set-theoretic applications, rather than on the applications themselves. Through this work a very attractive picture of the universe V of sets is starting to emerge. We shall discuss 7 aspects of this picture, mentioning some of the important open questions that remain.

1. Constructibility.

Set-theorists have had great success in solving problems under the hypothesis $V = L$. Under this assumption, Gödel proved the Generalised Continuum Hypothesis and also precisely determined the behaviour of projective sets of reals in L, with regard to regularity properties such as Lebesgue measurabilty and the perfect set property. The deepest work on L was accomplished by Jensen:

Suslin's Hypothesis in L. ($\diamondsuit$ Principle)
The Generalised Suslin Hypothesis in L. ($\square$ Principle)
Higher-Gap Transfer in L. (Morasses)

Important work remains to be done on Higher-Gap Transfer. The Gap n Transfer Theorem states that if a structure of size $\kappa^{(+n)}$ for a countable language carries a unary predicate of size κ, κ infinite, then for any infinite cardinal λ, there is an elementarily equivalent structure of size $\lambda^{(+n)}$ whose unary predicate has size λ. Jensen produced a proof in L for regular λ, through the use of a Gap $(n-1)$ Morass. An attractive problem is to complete Jensen's work to the case of singular λ, and to isolate the morass-like principle required.

2. Forcing over L.

$V = L$ is not a theorem of ZFC: The forcing method allows us to consistently enlarge L to models $L[G]$ where G is a set or class that is generic over L with respect to some forcing notion P. Thus it is reasonable to suggest that V, rather than equal to L, should be in fact a generic extension $L[G]$ of L. But this then gives rise to the new and difficult question: Which generic extension is it? Usually, a forcing notion P gives rise not to one, but to many different generics G. Some hope is provided by:

Theorem 1. Assume some weak large cardinal axioms, consistent with $V = L$ (precisely: an n-ineffable cardinal for each n). Then there is an L-definable forcing notion P with at most one generic.

(Unless otherwise stated, we take "definable" to mean "definable *without* parameters".) But this does not solve our problem, for P is not the only forcing notion satisfying this Theorem and there does not seem to be a canonical choice for P. Moreover, the hypothesis that generics exist for all definable forcing notions is inconsistent:

Theorem 2. There exist forcing notions P_0, P_1 which are definable over L and which preserve ZFC, such that there cannot be generics for P_0 and P_1 simultaneously.

So if we want V to not be L we must decide for which forcing notions P to allow generics. The needed criterion arises naturally through the consideration of *CUB-absoluteness*:

Definition. A class C of ordinals is CUB iff it is closed and unbounded. V is *CUB-absolute over* L iff every L-definable class of ordinals which has a CUB subclass definable with parameters in a generic extension of V has one definable with parameters in V.

Theorem 3. V is CUB-absolute over L iff $0^\#$ exists.

$0^\#$ is a special set of integers discovered by Silver and Solovay, whose existence is a "transcendence principle for L" in the sense that it implies that V is not a generic extension of L. The existence of $0^\#$ is equivalent to the existence of a nontrivial elementary embedding of L into itself. If $0^\#$ exists then there is a smallest inner model which satisfies "$0^\#$ exists", namely the canonical model $L[0^\#]$.

Thus as an alternative to $V = L$ we could propose the hypothesis: $0^\#$ exists and $V = L[0^\#]$. This allows for the existence of generic extensions of L. Moreover we can now provide a generic existence criterion for L-definable forcings, by declaring an L-definable forcing to have a generic iff it has one definable in $L[0^\#]$.

An open problem is to provide a more convincing characterisation of $0^\#$ in terms of forcing. One possibility is suggested by the following

Theorem 4. Assume a weak large cardinal axiom, consistent with $V = L$ (precisely: an $\omega + \omega$-Erdős cardinal). If $0^\#$ exists and an L-definable forcing has a generic, then it has one definable in $L[0^\#]$.

Thus $L[0^\#]$ is "saturated" with respect to L-definable forcings. A nice result would be the converse to this, giving that $0^\#$ exists iff V is saturated with respect to L-definable forcings.

A second possibility would be to define a new concept of forcing and prove that the existence of $0^\#$ is equivalent to the statement that V is not "generic" over L in this new sense. One cannot simply use the usual notion of class forcing for this purpose; indeed there exist reals R in $L[0^\#]$ which are not class-generic over L and from which $0^\#$ is not constructible.

3. Large Cardinals.

If we are willing to accept the existence of $0^\#$, then it is difficult to not also admit the existence of $0^{\#\#}$, which relates to the model $L[0^\#]$ in the same way as $0^\#$ relates to L. Indeed, through iteration of a suitable "# operation", we are led to models much larger than L, which satisfy strong large cardinal axioms.

We have said that the existence of $0^\#$ is equivalent to the non-rigidity of L, i.e., to the existence of a nontrivial elementary embedding from L to itself. Let us use this as a basis for generalisation. Suppose that M is a non-rigid inner model and let $\pi : M \to M$ be a nontrivial elementary embedding from M to itself. Then $\pi(x) \neq x$ for some x and in fact for some least ordinal κ, called the *critical point of* π, $\pi(\kappa) \neq \kappa$. It can be shown that for α less than $(\kappa^+)^M$, the restriction $\pi \restriction \alpha$ is an element of M. We define the # (or *extender*) *derived from* π to be the restriction $E_\pi = \pi \restriction (\kappa^+)^M$. A # *for* M is a # derived from some nontrivial $\pi : M \to M$. Thus M has a # iff M is non-rigid.

A # *iteration* is a sequence $M_0, M_1, \ldots$ of inner models where
$M_0 = L$
$M_{i+1} = M_i[E_i]$, where E_i is a # for M_i
M_λ for limit λ is the "limit" of $\langle M_i \mid i < \lambda \rangle$.

The type of model that arises through such an iteration is called an *extender model* and is of the form $L[E]$ where $E = \langle E_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ is a sequence of extenders (for appropriate models). Many of the large cardinal notions, such as measurability, strength, Woodinness and beyond, have been shown to "reflect" to extender models, in the sense that if κ has one of these properties in V then it also has the same property in some extender model. These extender models are not built through #-iteration, but rather through an explicit choice of the extenders that make up the model.

The relevance of #-iterations to large cardinal theory derives from the following result.

Theorem 5. (joint with Koepke) Any of the known extender models for large cardinals result from a #-iteration.

How large an extender model can we produce through #-iteration? A #-iteration is *maximal* if it cannot be continued so as to produce a larger extender model. An extender model is *maximal* if it is the final model of a maximal #-iteration. Now there are two ways in which an extender model M can be maximal: Either there is no # for M, i.e. M is rigid, or M is not enlarged through the addition of a # for itself. The latter possibility leads us to a very strong hypothesis.

Theorem 6. Suppose that M is a non-rigid, maximal extender model. Then in M there is a superstrong cardinal.

Superstrength is much stronger than any of the large cardinal properties for which extender models have been constructed. It is equivalent to the existence of a nontrivial elementary embedding π of V into some inner model M such that M contains all bounded subsets of $\pi(\kappa)$, where κ is the critical point of π.

We have said nothing about the existence of maximal extender models. It can be shown that such models must exist, unless there is an inner model with a superstrong cardinal. Thus a plausible strategy for constructing an inner model with a superstrong cardinal is to build a maximal extender model, and then argue for its non-rigidity. We shall discuss this strategy further in Section 7 below.

A central open problem is to construct a maximal #-iteration whose resulting maximal extender model M satisfies L-like internal properties, such as GCH, $\square$ and the existence of Morasses. In addition one would like that models occuring in this maximal #-iteration before the final model M satisfy analogues of Theorems 3 and 4, relating a # for L to forcing over L. We discuss these properties next.

4. The Fine Structure of Extender Models.

A key result that lifts Jensen's work on the fine structure of L to inner models for large cardinals is the following.

Theorem 7. (Schimmerling-Zeman) $\square$ holds in the known extender models for large cardinals.

An open problem is to construct Morasses in extender models. Recent work of Zeman suggests that this is possible, using "canonical extender fragments".

5. Forcing over Extender Models.

An important technique needed to relate $0^\#$ to forcing over L is the method of *Jensen coding*, which enables one to L-code a class A of ordinals by a real R, in the sense that A is definable in $L[R]$. Jensen applied his coding method to prove that there are reals R in $L[0^\#]$ which are not set-generic over L, yet from which $0^\#$ is not constructible. In the context of extender models, one would like to prove a similar result, with the added requirement that large cardinal properties be preserved. A result of Woodin says that in a certain sense this is not possible in the presence of a Woodin cardinal. However one does have the following, which implies that Woodin's result is optimal:

Theorem 8. Suppose that M is one of the known extender models. Then there is a real (in a generic extension of M) which is not set-generic over M and which preserves any specified "degree of Woodinness".

An open problem is to fully develop Jensen's coding method in the extender model context, by showing that in the absence of Woodin cardinals, classes can be coded by reals preserving large cardinal properties. Another challenging problem is to develop the proper analogue of Theorem 4 concerning "generic saturation" in the extender model context. Analogues of Theorem 3 will be discussed in section 7 below.

6. Methods for Descriptive Set Theory.

An important application of large cardinal theory is to *descriptive set theory*. The central objects of study in this theory are the projective sets of reals, obtained from Borel sets by applying the operations of continuous image and complement. A fundamental result is Moschovakis' generalisation of the classical Kondo Uniformisation Theorem, stating that a binary co-analytic relation contains a co-analytic function with the same domain.

Moschovakis proved his theorem under the hypothesis of projective determinacy, a property that was later shown by Martin and Steel to be a consequence of large cardinal axioms. Recently, a direct proof of Moschovakis' Theorem from large cardinal theory was found by Neeman. He proved uniformisation for a wide variety of definability notions expressed in terms of extender models. Earlier work of Woodin, not based on determinacy, shows that the central definability notions of descriptive set-theory are covered by Neeman's Theorem, yielding Moschovakis' result as a corollary. An interesting open problem is to determine the exact strength of Moschovakis' Theorem (in the presence of regularity properties such as Lebesgue measurability and the Baire property for projective sets), in terms of large cardinals.

7. Axioms of Absoluteness.

We have described a picture of V in which L is enlarged through #-iteration to an extender model with a superstrong cardinal. But how do we motivate the claim that our #-iteration is long enough to reach such extender models? We have seen that a superstrong cardinal results from the non-rigidity of a maximal extender model. But how can we argue that maximal extender models must be non-rigid, in the way that we argued this for L?

This may be possible using further axioms of absoluteness. Suppose that M is an inner model of V. We define a class of ordinals C to be CUB$^+$ iff for α in a CUB class, $C \cap \alpha^+$ is CUB in α^+. There is a corresponding notion of *CUB$^+$-completeness* of V over M. The non-rigidity of the known extender models M can be proved using this notion, using the fact that Jensen's $\square$ principle is true in such models. This suggests that the non-rigidity of a maximal extender model can be established in this way, provided one has $\square$ in such a model. An alternative approach can be used if we work in a sufficiently strong class theory: Say that V is *definably-complete over M* iff every (amenable) class of M is definable with parameters in V. Again the non-rigidity of the known extender models M can be proved, but now without need for $\square$. As large cardinal properties slightly stronger then superstrength imply the failure of $\Box$, definable-completeness may be the preferred absoluteness principle.

However at the consistency level of *supercompactness*, far beyond superstrength, it is likely that definable-completeness can hold with respect to HOD, the inner model of hereditarily ordinal-definable sets. This raises the intriguing open question as to whether in fact rigidity can fail for HOD, at the level of supercompactness. A positive answer would seem to imply the impossibility of a reasonable inner model theory for a supercompact cardinal.