$\underset{\sim}{\delta}^1_2$ Without Sharps

Sy D. Friedman,*
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139 USA
e-mail: sdf@math.mit.edu

W. Hugh Woodin, UC Berkeley
Department of Mathematics
University of California
Berkeley, CA 94720 USA
e-mail: woodin@math.berkeley.edu

$\underset{\sim}{\delta}^1_2$ denotes the supremum of the lengths of $\underset{\sim}{\Delta}^1_2$ prewellorderings of the reals. A result of Kunen and Martin (see Martin[77]) states that $\underset{\sim}{\delta}^1_2$ is at most ω_2 and it is known that in the presence of sharps the assumption $\underset{\sim}{\delta}^1_2 = \omega_2$ is strong: it implies the consistency of a strong cardinal (see Steel-Welch[?]).

In this paper we show how to obtain the consistency of $\underset{\sim}{\delta}^1_2 = \omega_2$ in the absence of sharps, without strong assumptions.

Theorem. *Assume the consistency of an inaccessible. Then it is consistent that $\underset{\sim}{\delta}^1_2 = \omega_2$ and ω_1 is inaccessible to reals (i.e., $\omega_1^{L[x]}$ is countable for each real x).*

The proof is obtained by combining the Δ_1-coding technique of Friedman-Velickovic [95] with the use of a product of Jensen codings of Friedman [94].

We begin with a description of the Δ_1-coding technique.

Definitions. Suppose x is a set, $\langle x, \epsilon\rangle$ satisfies the axiom of extensionality and $A \subseteq \mathrm{ORD}$. *x preserves A* if $\langle x, A\cap x\rangle \cong \langle \bar{x}, A \cap \bar{x}\rangle$ where $\bar{x}$ = transitive collapse of x. For any ordinal δ, $x[\delta] = \{f(\gamma) | \gamma < \delta, f \in x, f$ a function, $\gamma \in \mathrm{Dom}(f)\}$. *$x$ strongly preserves A* if $x[\delta]$ preserves A for every cardinal δ. A sequence $x_0 \subseteq x_1 \subseteq \ldots$ is *tight* if it is continuous and for each i,

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$\langle \bar{x}_j | j < i \rangle$ belongs to the least ZF^--model which contains $\bar{x}_i$ as an element and correctly computes card$(\bar{x}_i)$.

Condensation Condition for A. Suppose t is transitive, δ is regular, $\delta \in t$ and $x \in t$. Then:

(a) There exists a continuous, tight δ-sequence $x_0 \prec x_1 \prec \cdots \prec t$ such that card$(x_i) = \delta$, $x \in x_0$ and x_i strongly preserves A, for each i.

(b) If δ is inaccessible then there exist x_i's as above but where card$(x_i) = \aleph_i$.

The following is proved in Friedman-Velickovic [95].

Δ_1-Coding. Suppose $V = L$ and the Condensation Condition holds for A. Then A is Δ_1 in a class-generic real R, preserving cardinals.

Now we are ready to begin the proof of the Theorem. Suppose κ is the least inaccessible and $V = L$. Let $\langle \alpha_i | i < \kappa^+ \rangle$ be the increasing list of all $\alpha \in (\kappa, \kappa^+)$ such that $L_\alpha =$ Skolem hull (κ) in L_α. For each $i < \kappa^+$ define $f_i : \kappa \longrightarrow \kappa$ by $f_i(\gamma) =$ ordertype $(ORD \cap$ Skolem hull (γ) in $L_{\alpha_i})$. By identifying f_i with its graph and using a pairing function we can think of f_i as a subset of κ. The following is straightforward.

Lemma 1. *Each f_i obeys the Condensation Condition. Indeed $\langle f_i | i < \kappa^+ \rangle$ jointly obeys the Condensation Condition in the following sense: Suppose t is transitive, δ is regular, $\delta \in t$, $x \in t$. Then their exists a tight δ-sequence $x_0 \prec x_1 \prec \cdots \prec t$ such that card$(x_i) = \delta, x \in x_0$ and each x_i strongly preserves all f_j for $j \in x_i$ (and if $\delta = \kappa$ then we can alternatively require card$(x_i) = \aleph_i$).*

Now, following Friedman [94] we use a "diagonally-supported" product of Jensen-style codings. For each $i < \kappa^+$ let $\mathcal{P}(i)$ be the forcing from Friedman-Velickovic [95] to make f_i Δ_1-definable in a class-generic real. Then $\mathcal{P}$ consists of all $p \in \prod_{i<\kappa^+} \mathcal{P}(i)$ such that for infinite ordinals γ, $\{i | p(i)(\gamma) \neq (\phi, \phi)\}$ has cardinality at most α and in addition $\{i | p(i)(0) \neq (\phi, \phi)\}$ is finite.

Now note that for successor cardinals $\gamma < \kappa$ the forcing $\mathcal{P}$ factors as $\mathcal{P}_\gamma * \mathcal{P}^{G_\gamma}$ where $\mathcal{P}_\gamma$ forces that $\mathcal{P}^{G_\gamma}$ has the γ^+-CC. Also the joint Condensation Condition of Lemma 1 implies that the argument of Theorem 3 of Friedman-

Velickovic [95] can be applied here to show that $\mathcal{P}_\gamma$ is $\leq \gamma$-distributive, and also that $\mathcal{P}$ is Δ-distributive (if $\langle D_i | i < \kappa \rangle$ is a sequence of predense sets then it is dense to reduce each D_i below $\aleph_{i+1}$). So $\mathcal{P}$ preserves cofinalities.

Thus in a cardinal-preserving forcing extension of L we have produced κ^+ reals $\langle R_i | i < \kappa^+ \rangle$ where R_i Δ_1-codes f_i and hence there are well-orderings of κ of any length $< \kappa^+$ which are Δ_1 in a real. Finally Lévy collapse to make $\kappa = \omega_1$ and we have $\underset{\sim}{\delta}^1_2 = \omega_2$, ω_1 inaccessible to reals. $\dashv$

The above proof also shows the following, which may be of independent interest.

Theorem 2. Let $\underset{\sim}{\delta}_1(\kappa)$ be the sup of the lengths of wellorderings of κ which are $\underset{\sim}{\Delta}_1$ over $L_\kappa[x]$ for some x, a bounded subset of κ. Then (relative to the consistency of an inaccessible) it is consistent that κ be weakly inaccessible and $\underset{\sim}{\delta}_1(\kappa) = \kappa^+$.

Remark. The conclusion of Theorem 2 cannot hold in the context of sharps: if κ is weakly inaccessible and every bounded subset of κ has a sharp then $\underset{\sim}{\delta}_1(\kappa) < \kappa^+$. This is because $\underset{\sim}{\delta}_1(\kappa)$ is then the second uniform indiscernible for bounded subsets of κ, which can be written as the direct limit of the second uniform indiscernible for subsets of δ, as δ ranges over cardinals less than κ; so $\underset{\sim}{\delta}_1(\kappa)$ has cardinality κ.

Using the least inner model closed under sharp, we can also obtain the following.

Theorem 3. Assuming it is consistent for every set to have a sharp, then this is also consistent with $\underset{\sim}{\delta}^1_3 = \omega_2$.

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