Definability Degrees

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March 22, 2005

We establish the equiconsistency of a simple statement in definability theory with the failure of the GCH at all infinite cardinals. The latter was shown by Foreman and Woodin ([2]) to be consistent, relative to the existence of large cardinals.

Definition. Suppose that κ is an infinite cardinal and A, B are subsets of κ. We say that A is *κ-definable from B*, written $A \leq_\kappa B$, iff A is definable over the structure $\langle L_\kappa[B], \in, B\rangle$. Two subsets of κ are *κ-incomparable* iff neither is κ-definable from the other.

We consider the following statement:

$(*)_\kappa$ For each $C \subseteq \kappa$ there are κ-incomparable $A, B \subseteq \kappa$ such that $C \leq_\kappa A$ and $C \leq_\kappa B$.

Theorem 0.1 *$(*)_\kappa$ holds for regular κ.*

Proof. Let C be a subset of κ and let P be κ-Cohen forcing over the ground model $L[C]$. If (A, B) is $P \times P$-generic over $L[C]$ then it is easy to verify that (A, C) and (B, C) are κ-incomparable. As only genericity over $L_{\kappa+1}[C]$ is required, there exist such A, B in V. □

*The author was supported by Project Number 16334-NO5 of the Austrian Sciene Fund (FWF).

Theorem 0.2 *Suppose that κ is a singular strong limit cardinal of uncountable cofinality and $(*)_\kappa$ holds. Then GCH fails at CUB-many $\bar\kappa < \kappa$.*

Proof. This proof comes from [3], which was based on [4]. Let $\langle \kappa_i \mid i < \gamma\rangle$ be a continuous cofinal sequence of cardinals less than κ of ordertype $\gamma = \mathrm{cof}\,\kappa > \omega$. Let $C \subseteq \kappa$ be chosen so that $H_\kappa = L_\kappa[C]$ and both the sequence of κ_i's as well as a wellordering $<$ of H_κ are κ-definable from C. Also assume that whenever the GCH holds at κ_i, all subsets of κ_i have $<$-rank less than κ_i^+.

For each ordinal $\delta < \kappa$ let c_δ be the $<$-least bijection between δ and its cardinality. For $\bar\delta < \delta$ set $c(\bar\delta, \delta) = c_\delta(\bar\delta)$. Suppose now that GCH holds at κ_i for each $i \in S$, where $S \subseteq \gamma$ is stationary. For each $A \subseteq \kappa$ define $f_A : S \to \kappa$ by:

$f_A(i) =$ the $<$-rank of $A \cap \kappa_i$.

Claim. Suppose that A, B are subsets of κ. If $f_A(i) \leq f_B(i)$ for stationary-many $i \in S$ then A is κ-definable from (B, C).

The Claim implies that $(*)_\kappa$ fails, as either $f_A(i) \leq f_B(i)$ for stationary-many $i \in S$ or $f_B(i) \leq f_A(i)$ for stationary-many $i \in S$.

Proof of Claim. Let S^* consist of all limit ordinals i in S such that $f_A(i) \leq f_B(i)$. Define the regressive function g on S^* by:

$g(i) =$ least j such that $c(f_A(i), f_B(i) + 1) < \kappa_j$.

By Fodor's Theorem, g is constant on a stationary, and hence unbounded, subset S^{**} of S^*. It follows that the function on S^{**}

$h(i) = c(f_A(i), f_B(i) + 1)$

has range bounded in κ and therefore is an element of $H_\kappa = L_\kappa[C]$. But for $i \in S^{**}$ we have

$f_A(i) = c^{-1}_{f_B(i)+1}(h(i))$

and therefore $f_A \restriction S^{**}$ is κ-definable from (B, C). It follows that A is also κ-definable from (B, C). $\square$

Theorem 0.3 *The following are equiconsistent:*
(a) $(*)_\kappa$ *holds for all* κ.
(b) GCH fails for all κ.

Proof. Actually (b) implies (a) directly: Suppose that $(*)_\kappa$ fails. Then for some $C \subseteq \kappa$, all initial segments of the prelinear ordering $\langle \{A \mid C \leq_\kappa A\}, \leq_\kappa \rangle$ have cardinality at most κ, and therefore $\{A \mid C \leq_\kappa A\}$ has cardinality at most κ^+. It follows that GCH holds at κ.

Now suppose that (a) holds; we show that (b) holds in a forcing extension of the universe. By Theorem 0.2, we know that GCH fails on a CUB subset of each singular strong limit cardinal of uncountable cofinality. It follows that the class X of cardinals where GCH fails is *very stationary*: For any CUB class D, $X \cap D$ contains arbitrarily-long closed subsets. Now add a CUB subclass C to X using conditions which are closed subsets of X, ordered by end-extension. Using the fact that X is very stationary, this forcing preserves ZFC (with a predicate for the generic class C) and adds no new sets (see [1]). GCH fails at cardinals in C. Now for each adjacent pair $\kappa < \lambda$ in C, add λ^{++} subsets of κ^+, using an Easton product. In the generic extension, cardinals are unchanged and GCH fails at cardinals outside of C. So we have created a model in which GCH fails at every infinite cardinal. $\square$

References

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