# David's Trick

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In David [D82] a method is introduced for creating reals $R$ which not only code classes in the sense of Jensen coding but in addition have the property that in $L[R]$, $R$ is the unique solution to a $\Pi^1_2$ formula. In this article we cast David's "trick" in a general form and describe some of its uses.

**Theorem.** *Suppose $A \subseteq \mathrm{ORD}$, $\langle L[A], A\rangle \models ZFC + 0^\#$ does not exist and suppose that for every infinite cardinal $\kappa$ of $L[A]$, $H^{L[A]}_\kappa = L_\kappa[A]$ and $\langle L_\kappa[A], A \cap \kappa\rangle \models \varphi$. Then there exists a $\Pi^1_2$ formula $\psi$ such that:*

*(a) If $R$ is a real satisfying $\psi$ then there is $A \subseteq \mathrm{ORD}$ as above, definable over $L[R]$ in the parameter $R$.*

*(b) For some tame, $\langle L[A], A\rangle$-definable, cofinality-preserving forcing $P$, $P \Vdash \exists R \psi(R)$.*

*Moreover if $A$ preserves indiscernibles then $\psi$ has a solution in $L[A, 0^\#]$, preserving indiscernibles.*

**Remark**

(1) We require that $H^{L[A]}_\kappa$ equal $L_\kappa[A]$ for infinite $L[A]$-cardinals solely to permit cofinality-preservation for $P$; if cofinality-preservation is dropped then such a requirement is unnecessary, by coding $A$ into $A^*$ with this requirement and then applying our result to $A^*$.

(2) A class $A$ *preserves indiscernibles* if the Silver indiscernibles are indiscernible for $\langle L[A], A\rangle$. It follows from the technique of Theorem 0.2 of

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Beller-Jensen-Welch [BJW82] (see Friedman [98]) that if $A$ preserves indiscernibles then $A$ is definable from a real $R \in L[A, 0^\#]$, preserving indiscernibles.

*Proof.* Our plan is to create an $\langle L[A], A\rangle$-definable, tame, cofinality-preserving forcing $P$ for adding a real $R$ such that whenever $L_\alpha[R] \models ZF^-$ there is $A_\alpha \subseteq \alpha$, definable over $L_\alpha[R]$ (via a definition independent of $\alpha$) such that $L_\alpha[R] \models$ for every infinite cardinal $\kappa$, $H_\kappa = L_\kappa[A_\kappa]$ and $\varphi$ is true in $\langle L_\kappa[A_\alpha], A_\alpha \cap \kappa\rangle$. This property $\psi$ of $R$ is $\Pi^1_2$ and gives us (a), (b) of the Theorem. The last statement of the Theorem will follow using Remark (2) above.

$P$ is obtained as a modification of the forcing from Friedman [97], used to prove Jensen's Coding Theorem (in the case where $0^\#$ does not exist in the ground model). The following definitions take place inside $L[A]$.

**Definition (Strings).** *Let $\alpha$ belong to* Card $=$ *the class of all infinite cardinals. $S_\alpha$ consists of all $s : [\alpha, |s|) \to 2$, $\alpha \le |s| < \alpha^+$ such that $|s|$ is a multiple of $\alpha$ and:*

*(a)* $\eta \le |s| \to L_\delta[A \cap \alpha, s \restriction \eta] \models \text{Card}\,\eta \le \alpha$ *for some* $\delta < (\eta^+)^L \cup \omega_2$.

*(b) If* $\mathcal{A} = \langle L_\beta[A \cap \alpha, s \restriction \eta], s \restriction \eta\rangle \models (ZF^-$ *and* $\eta = \alpha^+)$ *then over* $\mathcal{A}$, $s \restriction \eta$ *codes a predicate* $A(s \restriction \eta, \beta) = A^* \subseteq \beta$ *such that* $A^* \cap \alpha = A \cap \alpha$ *and for every cardinal* $\kappa$ *of* $L_\beta[A^*]$, $H_\kappa^{L_\beta[A^*]} = L_\kappa[A^*]$ *and* $\langle L_\kappa[A^*], A^* \cap \kappa\rangle \models \varphi$.

**Remark** When in (b) above we say that $s \restriction \eta$ *codes* $A^*$ we are referring to the canonical coding from the proof of Theorem 4 of Friedman [97] of a subset of $\beta$ by a subset of $(\alpha^+)^{\mathcal{A}} = \eta$ (relative to $A \cap \alpha$).

The remainder of the definitions from the proof of Theorem 4 of Friedman [97] remain the same in the present context. We now verify that he proofs of the lemmas from Friedman [97] can successfully accommodate the new restriction (clause (b)) on elements of $S_\alpha$.

**Lemma 1 (Distributivity for $R^s$).** *Suppose $\alpha \in$ Card, $s \in S_{\alpha^+}$. Then $R^s$ is $\alpha^+$-distributive in $\mathcal{A}^s$.*

*Proof.* Proceed as in the proof of Lemma 5 of Friedman [97]. The only new point is to verify that in the proof of the Claim, $t_\lambda$ satisfies clause (b) (of the new definition of $S_\alpha$). The fact that $s$ belongs to $S_{\alpha^+}$ and that $t_\lambda$ codes $\bar{H}_\lambda$ imply that clause (b) holds for $t_\lambda$ whenever $\beta$ is at most $\bar{\mu}_\lambda =$ the height of $\bar{H}_\lambda$. But as $|t_\lambda|$ is definably singular over $L_{\bar{\mu}_\lambda}[t_\lambda]$ these are the only $\beta$'s that concern us. □

**Lemma 2 (Extendibility of $P^s$).** *Suppose $p \in P^s$, $s \in S_\alpha$, $X \subseteq \alpha$, $X \in \mathcal{A}^s$. Then there exists $q \leq p$ such that $X \cap \beta \in \mathcal{A}^{q_\beta}$ for each $\beta \in \mathrm{Card} \cap \alpha$.*

*Proof.* Proceed as in the proof of Lemma 6 of Friedman [97]. In the definition of $q$, the only instances of clause (b) to check are for $s_\beta$ when Even $(Y \cap \beta)$ codes $s_\beta$, $s_\beta$ satisfying clause (a) of the definition of membership in $S_\beta$. But the embedding $\bar{\mathcal{A}}_\beta \to \mathcal{A}$ is $\Sigma_1$-elementary and instances of clause (b) refer to ordinals less than the height of $\mathcal{A}$; so the fact that $s$ belongs to $S_\alpha$ implies that $s_\beta$ belongs to $S_\beta$. □

**Lemma 3 (Distributivity for $P^s$).** *Suppose $s \in S_{\beta^+}$, $\beta \in \mathrm{Card}$.*

*(a) If $\langle D_i \mid i < \beta \rangle \in \mathcal{A}^s$, $D_i$ $i^+$ dense on $P^s$ for each $i < \beta$ and $p \in P^s$ then there is $q \leq p$, $q$ meets each $D_i$.*

*(b) If $p \in P^s$, $f$ small in $\mathcal{A}^s$ then there exists $q \leq p$, $q \in \Sigma_f^p$.*

*Proof.* Proceed as in the proof of Lemma 7 of Friedman [97]. In the Claim we must verify that $p_\gamma^\lambda$ satisfies clause (b). But once again this is clear by the $\Sigma_1$-elementary of $\bar{H}_\lambda(\gamma)$ and the n fact that $L_{\bar{\mu}}[A \cap \gamma, p_\gamma^\lambda] \models |p_\gamma^\lambda|$ is $\Sigma_1$-singular, where $\bar{\mu} =$ height of $\bar{H}_\lambda(\gamma)$. □

The argument of the proof of Lemma 3 can also be applied to prove the distributivity of $P$, observing that when building sequences of conditions $\langle p^i \mid i < \lambda \rangle$, $\lambda$ limit to meet an $\langle L[A], A \rangle$-definable sequence of dense classes, one has that $p_\gamma^\lambda$ codes $\bar{H}^\lambda(\gamma)$ of height $\bar{\mu}$, where $L_{\bar{\mu}+1}[A \cap \gamma, p_\gamma^\lambda] \models |p_\gamma^\lambda|$ is not a cardinal. Thus there is no additional instance of clause (b) to verify beyond those considered in the proof of Lemma 3.

Thus $P$ is tame and cofinality-preserving. The final statement of the Theorem also follows, using Remark (2) immediately after the statement of the Theorem. □

## Applications

(1) Local $\Pi^1_2$-Singletons. David [D82] proves the following: There is an $L$-definable forcing $P$ for adding a real $R$ such that $R$ is a $\Pi^1_2$-singleton in every set-generic extension of $L[R]$ (via a $\Pi^1_2$ formula independent of the set-generic extension). This is accomplished as follows: One can produce an $L$-definable sequence $\langle T(\kappa) \mid \kappa$ an infinite $L$-cardinal$\rangle$ such that $T(\kappa)$ is a $\kappa^{++}$-Suslin tree in $L$ for each $\kappa$ and the forcing $\prod T(\kappa)$ for adding a branch $b(\kappa)$ through each $T(\kappa)$ (via product forcing, with Easton support) is tame and cofinality-preserving. Now for each $n$ let $X_n \subseteq \omega_1^L$ be class-generic over $L$, $X_n$ codes a branch through $T(\kappa)$ iff $\kappa$ is of the form $(\aleph^L_{\lambda+n})$, $\lambda$ limit. The forcing $\prod P_n$, where $P_n$ adds $X_n$, can be shown to be tame and cofinality-preserving. Finally over $L[\langle X_n \mid n \in \omega\rangle]$ add a real $R$ such that $n \in R$ iff $R$ codes $X_n$. Then one has that in $L[R]$, $n \in R$ iff $T(\aleph^L_{\lambda+n})$ is not $\aleph^L_{\lambda+n}$-Suslin for sufficiently large $\lambda$. Clearly this characterization will still hold in any set-generic extension of $L[R]$. David's trick is used to strengthen this to a $\Pi^1_2$ property of $R$.

(2) A Global $\Pi^1_2$-Singleton. Friedman [90] produces a $\Pi^1_2$-singleton $R$, $0 <_L R <_L 0^\#$. This is accomplished as follows: assume that one has an index for a $\Sigma_1(L)$ classification $(\alpha_1 \cdots \alpha_n) \mapsto r(\alpha_1 \cdots \alpha_n)$ that produces $r(\alpha_1 \cdots \alpha_n) \in 2^{<\omega}$ for each $\alpha_1 < \cdots < \alpha_n$ in ORD such that $R = \cup\{r(i_1 \cdots i_n) \mid i_1 < \cdots < i_n$ in $I =$ Silver indiscernibles $\}$. For each $r \in 2^{<\omega}$ there is a forcing $\mathbb{Q}(r)$ for "killing" all $(\alpha_1 \cdots \alpha_n)$ such that $r(\alpha_1 \cdots \alpha_n)$ is incompatible with $r$. No $(i_1 \cdots i_n)$ from $I^n$ can be killed. Now build $R$ such that $r \subseteq R$ iff $R$ codes a $\mathbb{Q}(r)$-generic. Then $R$ is the unique real with this property. David's trick is used to strengthen this to a $\Pi^1_2$ property.

(3) New $\Sigma^1_3$ facts. Friedman [98] shows that if $M$ is an inner model of ZFC, $0^\# \notin M$, then there is a $\Sigma^1_3$ sentence false in $M$ yet true in a forcing extension of $M$. This is accomplished as follows: let $\langle C_\alpha | \alpha$ $L$-singular$\rangle$ be a $\Box$-sequence in $L$; i.e., $C_\alpha$ is CUB in $\alpha$, $otC_\alpha < \alpha$, $\bar{\alpha} \in \lim C_\alpha \to C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$. Define $n(\alpha) = 0$ if $otC_\alpha$ is $L$-regular and otherwise $n(\alpha) = n(otC_\alpha) + 1$. Then for some $n$, $\{\alpha \mid n(\alpha) = n\}$ is stationary in $M$. And for each $n$, there is a tame forcing extension of $M$ in which $\{\alpha \mid n(\alpha) \leq n\}$ is non-stationary,

and is in fact disjoint from the class of limit cardinals. David's trick is used to strengthen the latter into a $\Sigma^1_3$ property.

## References


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