ESI Workshop

ESI = Erwin Schrödinger Institut, Vienna

ESI WORKSHOP ON LARGE CARDINALS AND DESCRIPTIVE SET THEORY
June 14–25, 2009

1st week: June 14–18 Emphasis on Large Cardinals
2nd week: June 21–25 Emphasis on Descriptive Set Theory

All are welcome; no registration fee

For further information:

KGRC Webpage: http://www.logic.univie.ac.at/

Contact person: Jakob Kellner
email: esi2009@logic.univie.ac.at

The Hyperuniverse and Gödel Maximality

What should the universe V of sets look like?

Many possibilities:

- L (Gödel's constructible universe)
  CH true
  Singular cardinal hypothesis true
  A definable, non-measurable set of reals
  Suslin's hypothesis false
  Whitehead conjecture false
  Borel conjecture false
  Borel-isomorphism of non-Borel analytic sets false
  Singular Square principle true

Interpretations of V

- $L[G]$'s (Cohen-style forcing extensions of L)
  CH true, or not!
  Singular cardinal hypothesis still true
  A definable non-measurable set of reals, or not!
  Suslin's hypothesis true, or not!
  Whitehead's conjecture true, or not!
  Borel conjecture true, or not!
  Borel-isomorphism of non-Borel analytic sets still false
  Singular Square principle still true

Interpretations of V

- Big enough K's (Jensen-style core models)
  CH true
  Singular cardinal hypothesis true
  No definable non-measurable set of reals!
  Suslin's hypothesis false
  Whitehead conjecture false
  Borel conjecture false
  Borel-isomorphism of non-Borel analytic sets true!
  Singular Square principle true

Intepretations of V

- $K[G]$'s (Forcing extensions of K)
  Singular cardinal hypothesis true, or not!
  Singular square principle true
- Models with very LARGE cardinals
  Singular square principle false!
- Models where Forcing Axioms hold
  CH false!
  Suslin's hypothesis true!
  Borel's conjecture true!
  Singular cardinal hypothesis true!

What an interesting mess!

Which universe should we pick?

Minimal and Maximal Universes

Two seductive pictures of V:

- Minimal one: $V = L$
- Maximal one: ???

Gödel and Scott

Gödel (1964):

”From an axiom in some sense opposite to [V=L], the negation of Cantor’s conjecture could perhaps be derived. I am thinking of an axiom which ... would state some maximum property of the system of all sets, whereas [V=L] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ...”

Gödel and Scott

Scott (1977):

"I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms; but the models are all just models of the first order axioms and first-order logic is weak. I still feel that it ought to be possible to have strong axioms which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute ... But really pleasant axioms have not been produced by someone else or me, and the suggestion remains speculation. A new idea (or point of view) is needed, and in the meantime all we can do is to study the great variety of models."

The Search for Maximal Universes

How do we find a Maximal Universe?

Problem: V has all sets, so V is trivially maximal

We need to *compare* V to other possible universes

How do we create other possible universes?

Fact. If V were countable, then we could create many other possible universes (by forcing, infinitary logic, ...)

Solution: We *temporarily* treat V as a *countable* universe, embedded into a collection of other possible such universes

The Hyperuniverse

(von Neumann-Zermelo) V is determined by:

- Its Ordinals Ord
- Its Power Set operation $\mathcal{P}$

  $V_0 = \emptyset$
  $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
  $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$

V is countable, so $\mathrm{Ord}(V) =$ some countable ordinal α

Fix α

$\mathcal{H} =$ the Hyperuniverse
$\mathcal{H} =$ *All* countable transitive models of ZFC of ordinal height α

Universe $=$ element of the Hyperuniverse

What is α? We will choose α so that there is a “maximal” Universe

The Search for Maximal Universes

V_0 is an *inner* universe of V_1 iff $V_0 \subseteq V_1$
V_0 is an *outer* universe of V_1 iff $V_1 \subseteq V_0$
V_0, V_1 are *compatible universes* iff
they have a common outer universe

Q. What does it mean for a universe to be "maximal"?

Maximal = Maximal under inclusion?
NO! For every universe there is a larger outer universe

Instead, use *truth in inner universes* to define maximality:

The Search for Maximal Universes

$\mathcal{L}$ = language of set theory
For a universe W:
$\Phi(W)$ = all sentences of $\mathcal{L}$ which are true
in some *inner universe* of W

Obviously: $V \subseteq W \to \Phi(V) \subseteq \Phi(W)$

Key Definition:
V is *maximal* iff $V \subseteq W \to \Phi(V) = \Phi(W)$

The *Inner Model Hypothesis* states:
The universe V is maximal

The Inner Model Hypothesis

Objection: V is not countable!

Three good replies:

- We only treated V as countable *temporarily*. The IMH only says that V should satisfy sentences which are true in countable, maximal universes.
- In the IMH, we could restrict to universes which are inner universes of "forcing extensions" of V; then the IMH is a principle of ordinary "class theory".
- Are you sure that V is not countable? :)
  Maybe we should just figure out which *countable* universes are the good ones.

The Inner Model Hypothesis

Is the IMH consistent?

Theorem

(F-Woodin) Assume that there is a Woodin cardinal and a larger inaccessible cardinal. Then there are maximal universes, so the IMH is consistent.

Are large cardinals necessary?

Theorem

(F-Welch) The IMH implies that there are inner models with measurable cardinals of arbitrarily high Mitchell order.

The Inner Model Hypothesis

In favour of the IMH

Suppose the IMH fails.
Then there is an outer universe W such that $\Phi(V) \subsetneqq \Phi(W)$.
I.e. for some statement φ:
φ holds in some inner universe of W but in no inner universe of V

But then V is not big enough; we should replace V by W!

The Inner Model Hypothesis

Against the IMH

1. Socio-Political problem: The IMH is too strong!

The IMH implies:

There are no large cardinals in V
(they exist only in inner universes of V)
$R^{\#}$ does not exist for some real R

Set-theorists *love* large cardinals and $\#$'s

What should we do?

The Inner Model Hypothesis

Option 1: Radical change of view

Large cardinals can exist in inner models, but not in V
Not so bad!

Option 2: A large cardinal compromise

The *Relativised IMH*

Let T be ZFC + large cardinals.
IMH relative to T: T holds in V and:
$V \subseteq W$, T holds in $W \rightarrow \Phi(V) = \Phi(W)$

But why assume T?

The Inner Model Hypothesis

2. Mathematical problem: The IMH is not strong enough!

The IMH implies:

Singular cardinal hypothesis true
A definable, non-measurable set of reals
Borel-isomorphism of non-Borel analytic sets false
Singular Square principle true

But:

V satisfies IMH, $V \subseteq W \to W$ satisfies IMH

So: IMH does not resolve the Continuum Problem

The Strong Inner Model Hypothesis

The Strong IMH

The Strong IMH = The IMH with *absolute* parameters

p is *totally absolute* iff some formula defines p in *all* outer universes

ω is totally absolute

Is $\aleph_1$ totally absolute?

Probably not: $V \subseteq W$ does not imply $\aleph_1^V = \aleph_1^W$

A cardinal κ is *absolute* iff some formula defines κ in all outer universes W with the same cardinals $\leq \kappa$

$\aleph_1$, $\aleph_{99}$, $\aleph_{\omega+1} \cdots$ are absolute

SIMH $\rightarrow c = 2^{\aleph_0}$ is *not* absolute

SIMH $\rightarrow c \neq \aleph_1, \aleph_2, \aleph_3, \cdots$ (strong negation of CH)

But is the SIMH consistent?

The Strong Inner Model Hypothesis

Theorem

Assuming the existence of a Woodin cardinal and a larger inaccessible cardinal, the SIMH is consistent for the parameter ω_1.

Conjecture: The SIMH is consistent relative to large cardinals.

"From an axiom in some sense opposite to [V=L], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which ... would state some maximum property of the system of all sets, whereas [V=L] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ..."

The Internal Consistency programme

The IMH: $\Phi(V)$ is maximal

$\Phi(V)$ = All sentences true in some inner universe

φ is *internally consistent* iff φ belongs to $\Phi(V)$, i.e.,
iff φ is true in some inner universe

But what if $V = L$? Then there is only one inner universe!

Assumption: There are inner universes of V with large cardinals

Internal Consistency

A new type of consistency result.

$\mathrm{Con}(\mathrm{ZFC} + \varphi) = \mathrm{ZFC} + \varphi$ is consistent

$\mathrm{ICon}(\mathrm{ZFC} + \varphi) = \mathrm{ZFC} + \varphi$ holds in some inner universe

Consistency result:
$\mathrm{Con}(\mathrm{ZFC} + \mathrm{LC}) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \varphi)$,
where LC is a large cardinal axiom

Internal consistency result:
$\mathrm{ICon}(\mathrm{ZFC} + \mathrm{LC}) \rightarrow \mathrm{ICon}(\mathrm{ZFC} + \varphi)$

Internal consistency is stronger than consistency

Proving Internal Consistency *demands new techniques* and leads to *new consistency results*

Examples of Internal Consistency

Some Internal Consistency Results

Cardinal Exponentiation: F-Ondrejović, F-Honzík

Costationarity of the Ground Model: Dobrinen-F

Global Domination: F-Thompson

Tree Property: Dobrinen-F

Embedding Complexity: F-Thompson

Cofinality of the Symmetric Group: F-Zdomskyy

Internal Consistency: Cardinal Exponentiation

Cardinal Exponentiation

Easton function:
F : Reg $\to$ Card, F nondecreasing, $\mathrm{cof}(F(\kappa)) > \kappa$ for all $\kappa \in$ Reg

Easton: F a cardinal absolute Easton function. Then
$\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + 2^\kappa = F(\kappa)$ for all regular $\kappa)$

Easton used an Easton product

This gives *no* internal consistency result

Internal Consistency: Cardinal Exponentiation

F-Ondrejović: Instead use Easton iteration of Easton products and *generic modification*

Theorem

F a cardinal absolute Easton function. Then
$ICon(ZFC + 0^{\#} \text{ exists}) \to ICon(ZFC + 2^{\kappa} = F(\kappa) \text{ for all regular } \kappa)$

New consistency result (F-Honzík): F a cardinal absolute Easton function, κ is $H(F(\kappa))$-hypermeasurable witnessed by j with $j(F)(\kappa) \geq F(\kappa)$. Then in an outer universe, κ is measurable and F is realised. Sample corollary:

Theorem

$Con(ZFC + \text{ There is an } H(\kappa^{+n}) \text{ hypermeasurable}) \to$
$Con(ZFC + 2^{\kappa} = \kappa^{+n} \text{ for all regular } \kappa + \text{ There is a measurable cardinal})$

Internal Consistency: Global Domination

Global Domination

κ an infinite regular cardinal
Suppose $f, g : \kappa \to \kappa$
f *dominates* g iff $f(\alpha) > g(\alpha)$ for sufficiently large $\alpha < \kappa$
$\mathcal{F}$ is a *dominating family* iff
every $g : \kappa \to \kappa$ is dominated by some f in $\mathcal{F}$
$d(\kappa) =$ the smallest cardinality of a dominating family

Fact: $\kappa < d(\kappa) \leq 2^\kappa$ for all infinite regular κ

Global Domination: $d(\kappa) < 2^\kappa$ for all infinite regular κ

Internal Consistency: Global Domination

Cummings-Shelah: Con(ZFC) $\rightarrow$ Con(ZFC+Global Domination)
Proof uses κ-Cohen and κ-Hechler forcings
Corollary to their proof:
ICon(ZFC+ a supercompact cardinal) $\rightarrow$
ICon(ZFC+ Global Domination)

F-Thompson: Instead use κ-Sacks product (and *tuning forks*)

Theorem

ICon($ZFC + 0^{\#}$ exists) $\rightarrow$ ICon(ZFC+ Global Domination)

New consistency result:

Theorem

Con($ZFC + \kappa$ is $H(\kappa^{++})$ hypermeasurable) $\rightarrow$ Con(ZFC+ Global domination holds and there is a measurable cardinal).

Internal Consistency: The Tree Property

The Tree Property

(κ regular) A κ-*Aronszajn tree* is a κ-tree with no κ-branch
κ has the *tree property* iff there is no κ-Aronszajn tree

Mitchell: Con(ZFC+ Proper class of weakly compact cardinals) $\rightarrow$ Con(ZFC+ There is a proper class of inaccessibles + α^{++} has the tree property for all inaccessible α)

Proof uses "Mitchell forcing"

Corollary to proof:
ICon(ZFC+ a supercompact cardinal) $\rightarrow$
ICon(ZFC+ There is a proper class of inaccessibles + α^{++} has the tree property for all inaccessible α)

Internal Consistency The Tree Property

Dobrinen-F: Instead use iterated κ-Sacks forcing

Theorem

$ICon(ZFC + 0^{\#}$ exists$) \rightarrow$
$ICon(ZFC+$ There is a proper class of inaccessibles $+\ \alpha^{++}$ has the tree property for all inaccessible $\alpha)$

Theorem

$Con(ZFC+$ There is a weakly compact hypermeasurable$) \leftrightarrow Con(ZFC+$ The tree property holds at κ^{++} for a measurable $\kappa)$.

Internal Consistency. The Tree Property

A related consistency result:

Foreman:
Con(ZFC$+$ supercompact $+$ a larger weak compact) $\rightarrow$
Con(ZFC$+$ Tree Property at λ^{++} for a singular λ)

Theorem

(F-Halilović-Magidor) Con$($*ZFC* $+\ \kappa$ *weakly compact hypermeasurable*$)$ $\rightarrow$ *Con*$($*ZFC*$+$ *Tree property at* $\aleph_{\omega+2})$

Internal Consistency: Embedding Complexity

Embedding Complexity

$\alpha \leq \kappa$ infinite and regular

$G(\alpha, \kappa)$ = Set of graphs of size κ which omit α-cliques

Embedding complexity of $G(\alpha, \kappa) = \mathrm{ECG}(\alpha, \kappa)$:
Smallest size of a $U \subseteq G(\alpha, \kappa)$ such that every graph in $G(\alpha, \kappa)$ embeds into some element of U (as a subgraph)

What are the possibilities for $\mathrm{ECG}(\alpha, \kappa)$ as a function of α and κ?

Internal Consistency: Embedding Complexity

Complexity triple (a, c, F):
a, c, F : Reg $\rightarrow$ Card
F is an Easton function
$a(\kappa) \leq \kappa < c(\kappa) \leq F(\kappa)$ for all κ

Theorem

(Džamonja-F-Thompson) Suppose that (a, c, F) is a cardinal absolute complexity triple. Then $Con(ZFC) \rightarrow Con(ZFC + ECG(a(\kappa), \kappa) = c(\kappa)$ and $2^\kappa = F(\kappa)$ for all $\kappa \in Reg)$

Internal Consistency: Embedding Complexity

Theorem

(F-Thompson) Suppose that (a, c, F) is a cardinal absolute complexity triple. Then $ICon(ZFC + 0^\# \text{ exists}) \rightarrow ICon(ZFC + ECG(a(\kappa), \kappa) = c(\kappa) \text{ and } 2^\kappa = F(\kappa) \text{ for all } \kappa \in Reg)$

The generic is built using *partial master conditions*

Consistency with measurability: Looks difficult. Need a "tree-like" forcing to control embedding complexity

Internal Consistency: Cofinality of the Symmetric Group

Cofinality of the Symmetric Group

κ regular.
$\mathrm{Sym}(\kappa)$ = the symmetric group on κ

$\mathrm{cof}(\mathrm{Sym}(\kappa))$ = the length of the shortest chain of proper subgroups of $\mathrm{Sym}(\kappa)$ whose union is all of $\mathrm{Sym}(\kappa)$

Sharp-Thomas: $\mathrm{Con(ZFC)} \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{cof}(\mathrm{Sym}(\kappa)) = \kappa^{++}$ for an uncountable, regular κ).
Uses Shelah's "uniformisation" forcing. Combined with the *partial master conditions* of F-Thompson:

Theorem

(F-Zdomskyy) $I\mathrm{Con}(ZFC + 0^{\#}$ exists$) \to$ $I\mathrm{Con}(ZFC + \mathrm{cof}(\mathrm{Sym}(\alpha)) = \alpha^{++}$ for all "even" regular cardinals α).

Internal Consistency: Cofinality of the Symmetric Group

cof($Sym(\kappa)$) for a measurable κ?
Using an uncountable version of Miller forcing:

Theorem

(F-Zdomskyy) $Con(ZFC + \kappa \text{ is } H(\kappa^{++}) \text{ hypermeasurable}) \rightarrow Con(ZFC + cof(Sym(\kappa)) = \kappa^{++} \text{ for a measurable } \kappa)$

Open Problems

What global patterns can be realised in inner models of $L[0^\#]$ for the following characteristics?
Easton functions with parameters
Dominating pairs (d, F)
Sym(κ)
Tree Property (κ)
Stationary reflection at κ
$\square_\kappa$

What global patterns can be realised consistently with the existence of large cardinals?

What is the consistency strength of the IMH? Is the SIMH consistent?