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CODING OVER A MEASURABLE CARDINAL

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The purpose of this paper is to extend the coding method (see Beller, Jensen and Welch [82]) into the context of large cardinals.

THEOREM. *Suppose μ is a normal measure on κ in V and $\langle V, A\rangle \models ZFC$. Then there is a $\langle V, A\rangle$-definable forcing $\mathscr{P}$ for producing a real R such that:*

(a) *$V[R] \models ZFC$ and A is $V[R]$-definable with parameter R.*

(b) *$V[R] = L[\mu^*, R]$, where μ^* is a normal measure on κ in $V[R]$ extending μ.*

(c) *$V \models GCH \rightarrow \mathscr{P}$ is cardinal and cofinality preserving.*

COROLLARY. *It is consistent that μ is a normal measure, $R \subseteq \omega$ is not set-generic over $L[\mu]$ and $0^\dagger \notin L[\mu, R]$.*

Some other corollaries will be discussed in §4 of the paper.

The main difficulty in $L[\mu]$-coding lies in the problem of "stationary restraint". As in all coding constructions, conditions will be of the form $p(\gamma) = (p_\gamma, \bar{p}_\gamma)$, γ belonging to an initial segment of the cardinals, where $p(\gamma)$ is a condition for almost disjoint coding $p_{\gamma^+}: |p_{\gamma^+}| \rightarrow 2$, $|p_{\gamma^+}| < \gamma^{++}$ into a subset of γ^+. In addition for limit cardinals γ in Domain(p), $\langle p_{\gamma'} \mid \gamma' < \gamma\rangle$ serves to code p_γ.

An important restriction in coding arguments is that for inaccessible γ, $\bar{p}_{\gamma'} \neq \varnothing$ for only a *nonstationary* set of $\gamma' < \gamma$. The reason is that otherwise there are conflicts between the restraint imposed by the different $\bar{p}_{\gamma'}$ and the need to code extensions of p_γ below γ.

However the natural approach to $L[\mu]$-coding violates such a restriction. In this approach, the key to preserving the measure is "μ-distributivity", a property which requires μ-measure 1 restraint. To state μ-distributivity, suppose that D_α is open dense for all $\alpha < \kappa$. Then μ-distributivity asserts that any p can be extended to q so that q meets D_α for all α in a set of μ-measure 1. This enables one to show that μ in fact generates a measure μ^* in the generic extension defined by: $\mu^*(X) = 1$ iff $\mu(Y) = 1$ for some $Y \subseteq X$.

Instead we establish a weakening of μ-distributivity, which is most easily described in terms of elementary embeddings. The μ-distributivity property implies that if G is $\mathscr{P}$-generic then $j[G]$ generates a $j(\mathscr{P})$-generic H over M, where

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$j: V \to M$ is the ultrapower via the measure μ. By "generates" we mean that $H = \{q \in j(\mathscr{P}) \mid j(p) \leq q \text{ for some } p \in G\}$. Our weakening states not that H is $j(\mathscr{P})$-generic but only that H^* is $j(\mathscr{P})$-generic, where $H^* = \{q \in j(\mathscr{P}) \mid q \text{ is compatible with every } j(p) \text{ for } p \in G\}$. (Conditions in H^* are obtained by introducing appropriate restraint at κ to conditions in H; note that $q \in H \to \bar{q}_\kappa = \varnothing$.)

Genericity for H^* is obtained by defining $\mathscr{P}$ to consist of coding conditions p with domain Card $\cap\ \kappa^+$, where a restraint $\bar{p}_\kappa$ is specified for coding $j(p)_{\kappa^+}$ into κ^+. Then generically we will have that G_κ codes $\bigcup\{j(p)_{\kappa^+} \mid p \in G\}$, which is the key to establishing the genericity of H^*. We must also mix this with the coding of $A \subseteq \kappa^+$ into G_κ via μ, where $H_\alpha = L_\alpha[A]$ for infinite cardinals α.

And there is the usual dose of fine structure, this time for $L[\mu]$ rather than L. As proofs of these facts are straightforward, we do not establish them here.

§1 of the paper discusses the successor coding. In §2 we then turn to the definition of the limit coding. In §3 we discuss the full forcing $\mathscr{P}$ and prove the theorem. §4 considers some corollaries and related results.

§1. The successor coding. By Jensen [68] we can assume in our theorem that $V \models$ GCH and hence can fix $A \subseteq$ ORD such that $H_\alpha = L_\alpha[A]$ for all infinite cardinals α, where $H_\alpha =$ all sets of hereditary cardinality less than α. Fix a cardinal $\alpha < \kappa$ and we now define S_α, a collection of "strings" $s: [\alpha, |s|) \to 2$, $|s| < \alpha^+$. For s to belong to S_α we require that s is "μ, $A \cap \alpha$-reshaped". This means that for $\eta \leq |s|$, $L[\mu, A \cap \alpha, s \restriction \eta] \models \text{card}(\eta) \leq \alpha$. The reshaping of s allows us to code s by a subset of α, in the manner which we now describe.

For $s \in S_\alpha$ define structures $\mathscr{A}^0_s = L_{\nu^0_s}[\mu, A \cap \alpha, s^*]$ and $\mathscr{A}_s = L_{\nu_s}[\mu, A \cap \alpha, s^*]$ as follows (where $s^* = \{\nu_{s\restriction\eta} \mid s(\eta) = 1\}$): If $|s| = \alpha$ then $\nu^0_s = \kappa$. In general, $\nu^0_s = \bigcup\{\nu_{s\restriction\eta} \mid \eta < |s|\}$ and $\nu_s =$ the least p.r. closed $\nu > \nu^0_s$ such that $L_\nu[\mu, A \cap \alpha, s^*] \models \text{card}(|s|) \leq \alpha$. Of course ν^0_s, ν_s and hence $\mathscr{A}^0_s$, $\mathscr{A}_s$ are well-defined due to the reshaping of s.

We must extend the definition of S_α to certain cases in which α is not a cardinal. Suppose that α is p.r. closed and $L_\alpha[A] \models$ there is a largest cardinal. Define $\tilde{S}_\alpha$ exactly as we defined S_α above and then define S_α to consist of all $s \in \tilde{S}_\alpha$ such that $\mathscr{A}^0_s \models \alpha$ is a cardinal. Thus it is possible that $\tilde{S}_\alpha$ has strings which cannot be properly extended in $\tilde{S}_\alpha$. For $s \in S_\alpha$ we write $\alpha(s) = \alpha$.

Using the fine structure of K we can show the following. Let $S^+ = \{s \mid s \in S_\alpha \text{ for some } \alpha, L_\alpha[A] \models \text{there is a largest cardinal}\}$.

Fact A. *There exists* $\langle C_s \mid s \in S^+ \rangle$ *such that*:

(a) C_s *is closed, unbounded in* ν^0_s, *and* o.t.$(C_s) \leq \alpha(s)$. *If* $|s|$ *is a successor ordinal then* o.t.$(C_s) = \omega$.

(b) $\nu \in \text{Lim}(C_s) \to \nu = \nu^0_{s\restriction\eta}$, $C_s \cap \nu = C_{s\restriction\eta}$ *for some* $\eta < |s|$.

(c) C_s *is uniformly definable over* $\mathscr{A}'_s$, *where* $\mathscr{A}'_s = L_{\nu'_s}[A \cap \alpha(s), \mu, s^*]$, $\nu'_s =$ *largest* ν *for which either* $L_\nu[A \cap \alpha(s), \mu, s^*] \models |s|$ *is a cardinal, or* $\nu = \nu^0_s$.

(d) *If* $\pi: \langle \bar{\mathscr{A}}, \bar{C} \rangle \xrightarrow{\Sigma_1} \langle \mathscr{A}^0_s, C_s \rangle$ *then, for some* $\bar{s} \in S^+$, $\pi = \tilde{\pi} \circ \sigma$ *where* $\tilde{\pi}: \langle \mathscr{A}^0_{\bar{s}}, C_{\bar{s}} \rangle \xrightarrow{\Sigma_1} \langle \mathscr{A}^0_s, C_s \rangle$ *and* σ *is an iteration of* $\langle \bar{\mathscr{A}}, \bar{C} \rangle$. *And* $\tilde{\pi}$ *extends uniquely to* $\tilde{\pi}': \mathscr{A}'_{\bar{s}} \xrightarrow{\Sigma_1} \mathscr{A}'_s$, $\tilde{\pi}'$ *sends* Σ_1 *projectum* $(\hat{\mathscr{A}}_{\bar{s}})$ *to* Σ_1 *projectum* $(\hat{\mathscr{A}}_s)$. ($\hat{\mathscr{A}}_s$ *is defined in the statement of Lemma* 1.2, *below.*)

The above form of $\square$ enables us to provide the correct definition of the K-quasimorass.

DEFINITION. For $\bar{s}, s \in S^+$ we have $\bar{s} < s$ if there exists π: $\langle \mathscr{A}^0_{\bar{s}}, C_{\bar{s}} \rangle \xrightarrow{Q} \langle \mathscr{A}^0_s, C_s \rangle$ where $\alpha(\bar{s}) =$ critical point (π). ($\xrightarrow{Q}$ means that the boundedness of Σ_1 predicates is preserved.)

The K-quasimorass has the following properties. Let $\bar{s} \leq s$ mean that $\bar{s} < s$ or $\bar{s} = s$. If $\bar{s} \leq s$ then $\pi_{\bar{s}s}$ denotes the unique π obeying the above definition, if $\bar{s} < s$; $\pi_{ss} = \mathrm{id} \restriction \langle \mathscr{A}^0_s, C_s \rangle$.

Fact B. (a) $<$ *is a tree and, for all* $s \in S^+$, $\{\alpha(\bar{s}) \mid \bar{s} < s\}$ *is closed in* $\alpha(s)$.

(b) *If* $\bar{t} \subseteq \bar{s} < s$ *then* $\bar{t} < \pi_{\bar{s}s}(\bar{t}) = t$ *and* $\pi_{\bar{t}t} = \pi_{\bar{s}s} \restriction \mathscr{A}^0_{\bar{t}}$.

(c) $\bar{s} < s \to (|\bar{s}|$ *is a limit iff* $|s|$ *is a limit).*

(d) $s\ a{<}$*-limit* $\to \mathscr{A}^0_s = \bigcup\{\mathrm{rng}(\pi_{\bar{s}s}) \mid \bar{s} < s\}$.

(e) s *not* $\subseteq$*-maximal in* $S_{\alpha(s)} \to s\ a\ <$*-limit.*

(f) $\bar{s} < s$, $\pi_{\bar{s}s}$ *noncofinal* $\to \pi_{\bar{s}s} = \pi_{\bar{s}t}$ *where* $t \subseteq s$, $v^0_t = \bigcup(\mathrm{Rng}(\pi_{\bar{s}s}) \cap \mathrm{ORD})$.

(g) *If* $\bar{s} <_* s$ ($\bar{s}$ *immediately* $<$*-precedes* s) *then define* $t \dashv s$ *if* $\bar{t} <_* t < \pi_{\bar{s}s}(\bar{t})$ *for some* $\bar{t} \subseteq \bar{s}$. *Then* $|s|$ *limit,* $\pi_{\bar{s}s}$ *cofinal* $\to \alpha(s) = \bigcup\{\alpha(t) \mid t \dashv s\}$.

We use the K-quasimorass to define the almost disjoint codes b_s, $s \in S^+$.

DEFINITION. For $s \in S^+$ let $s^+ = s * 0$. Then $b_s = \{\bar{s} \mid \bar{s}^+ < s^+\}$ when $s \neq \phi$; b_ϕ consists of all $\phi_\alpha \in S_\alpha$ where $\phi^+_\alpha < \phi^+$ and o.t.$\{\alpha' < \alpha \mid \phi^+_{\alpha'} < \phi^+\}$ is even.

Note that b_s as defined here is not a set of ordinals, but instead a set of strings. This reflects a modification of the usual almost disjoint coding: roughly we will have that for R^s-generic $D \subseteq \alpha(s)$, $s(\eta) = i$ ($i = 0$ or 1) iff arbitrarily large $\bar{s} * i, \bar{s} \in b_{s \restriction \eta}$, lie on D. Thus we do not control a final segment of $\{\mu_{\bar{s}} \mid \bar{s} \in b_{s \restriction \eta}\}$, only a final segment of $\{\bar{s} * i \mid \bar{s} \in b_{s \restriction \eta}, \bar{s}$ lying on $D\}$. This is a weaker restraint.

Suppose $s \in S_{\alpha^+}$, α a cardinal less than κ. In the usual definition of $R^s \subseteq \mathscr{A}_s$ we use conditions $(u, \bar{u})$ for certain u: $[\alpha, |u|) \to 2$, $|u| < \alpha^+$ and $\bar{u} \subseteq \{b_{s \restriction \eta} \mid s(\eta) = 0\}$, $\mathscr{A}_s$-card$(\bar{u}) \leq \alpha$. Here we restrict ourselves to a special dense subset of this collection of conditions. For $(u, \bar{u})$ to belong to R^s we insist that:

(a) u: $[\alpha, |u|) \to 2$, $u \in S_\alpha$.

(b) $\bar{u} = \{b_{s \restriction \eta} \mid s(\eta) = 0, s \restriction \eta \in \mathrm{Rng}(\pi(\bar{u}))\}$, where $\pi(\bar{u})$ is of the form $\pi_{d(\bar{u}), r(\bar{u})}$, $\pi(\bar{u})$ is cofinal and $r(\bar{u}) \subseteq s$.

(c) $d(\bar{u}) \in \mathscr{A}_u$.

Extension is defined as follows, where "t lies on v" means that $v \restriction \mathrm{Dom}(t) = t$ and $Z_\eta = \{\langle \eta, \gamma \rangle \mid \gamma \in \mathrm{ORD}\}$, $(u)_j(\gamma) = u(\langle j, \gamma \rangle)$.

(d) $(u', \bar{u}') \leq (u, \bar{u})$ if $u \subseteq u'$, $\bar{u} \subseteq \bar{u}'$, ($t \in b \in \bar{u}$, t lies on $(u')_1 \to t$ lies on $(u)_1$ or $t * 1$ does not lie on $(u')_1$) and ($\eta \in |u| - A$, $\delta \in Z_\eta$, $(u')_2(\delta) = 1 \to (u)_2(\delta) = 1$).

Thus we have that if D: $[\alpha, \alpha^+) \to 2$ is R^s-generic then D is reshaped, $(D)_1$ codes s by $s(\eta) = i$ iff $\bar{s} * i$ lies on $(D)_1$ for arbitrarily large $\bar{s} \in b_{s \restriction \eta}$, and $(D)_2$ codes $A \cap \alpha^+$ by $\eta \in A \cap \alpha^+$ iff $(D)_2 \cap Z_\eta$ is unbounded in α^+.

LEMMA 1.1. *Suppose* G *is* R^s*-generic over* $\mathscr{A}_s$ *and let* $D = \bigcup\{u \mid (u, \bar{u}) \in G\}$. *Then* $A \cap \alpha^+, s \in L_{\nu_s}[A \cap \alpha, \mu, D]$.

PROOF. Clear. $\dashv$

LEMMA 1.2. $R^{<s} = \bigcup\{R^t \mid t \subseteq s, t \neq s\}$ *has the* α^{++}*-CC in* $\hat{\mathscr{A}}_s = L_{\hat{\eta}_s}[A \cap \alpha^+, \mu, s^*]$, *where* $\hat{\nu}_s =$ *largest p.r. closed* ν *such that* $L_\nu[A \cap \alpha^+, \mu, s] \models |s|$ *is a cardinal or* $\nu = \nu^0_s$.

PROOF. It suffices to show that $(u_0, \bar{u}_0)$, $(u_1, \bar{u}_1)$ incompatible $\to u_0 \neq u_1$ or $d(\bar{u}_0) \neq d(\bar{u}_1)$. But this is clear as otherwise we can amalgamate $\bar{u}_0, \bar{u}_1$ to get $\bar{u}$ and extend $u_0 = u_1$ to u so that $d(\bar{u}) \in \mathscr{A}_u$, obtaining thereby the compatibility of $(u_0, \bar{u}_0)$, $(u_1, \bar{u}_1)$. $\dashv$

LEMMA 1.3. *R^s is $\leq\alpha$-distributive in $\mathscr{A}_s$.*

PROOF. R^s is equivalent to $(R_2 \times R_1) * R_0^E$ where if D is R^s-generic then $(D)_2$ is R_2-generic, $(D)_1$ is R_1-generic and $(D)_0$ is R_0^E-generic, $E = \{\langle i, \delta\rangle \in D \mid i > 0\} = D - Z_0$. The forcing R_2 simply adds a code for $A \cap \alpha^+$ and is easily seen to be $\leq\alpha$-closed. A condition in R_1 consists of $(u)_1 : [\alpha, |(u)_1|) \to 2$ together with $\pi = \pi_{d,r}$, $d \leq r \subseteq s$. To extend we have $(u')_1 \supseteq (u)_1$, $\mathrm{Rng}(\pi') \supseteq \mathrm{Rng}(\pi)$, $(u')_1 - (u)_1$ must avoid $b_{s\restriction\eta}$ if $\pi(d \restriction \bar{\eta}) = s \restriction \eta$. The only reason that R_1 is not obviously $\leq\alpha$-closed is that $\bigcup\{\mathrm{Rng}(\pi_i) \mid i < \lambda\}$ is not necessarily of the form $\mathrm{Rng}(\pi)$ for some π. But $\bigcup\{\mathrm{Rng}(\pi_i) \mid i < \lambda\} \subseteq \mathrm{Rng}(\pi)$ for some π, which is enough to establish $\leq\alpha$-closure for R_1.

Lastly the forcing R_0^E, viewed as a forcing over $\mathscr{A}_s[E] = L_{\nu_s}[E, \mu]$, is seen to be $\leq\alpha$-distributive as follows. Given $(u)_0 \in R_0^E$ and predense $\langle D_i \mid i < \alpha\rangle \in L_{\nu_s}[\mu, E]$, let $M_0 <_{\Sigma_1} M_1 <_{\Sigma_1} \cdots$ be the first $\alpha + 1$ Σ_1-elementary submodels $M_i <_{\Sigma_1} L_{\nu_s}[\mu, E]$ such that $\alpha \cup \{(u)_0, \langle D_i \mid i < \alpha\rangle, \mu, E\} \subseteq M_0$ and $M_i \cap \alpha^+ \in M_{i+1}$, $M_\lambda = \bigcup\{M_i \mid i < \lambda\}$ for limit $\lambda \leq \alpha$. Then choose $(u_i)_0 \in M_{i+1}$ so that $(u_0)_0 = (u)_0$ and $(u_{i+1})_0 \supseteq (u_i)_0$ meets D_i, $(u_\lambda)_0 = \bigcup\{(u_i)_0 \mid i < \lambda\}$ for limit $\lambda \leq \alpha$. We must show that $(u_\lambda)_0$ is well-defined at limit stages λ; that is, we must show that $L[\mu, E \cap \alpha_\lambda, (u_\lambda)_0] \models \mathrm{card}(\alpha_\lambda) \leq \alpha$, where $\alpha_\lambda = |(u_\lambda)_0|$. Clearly α_λ is π_1-singular over $L_{\bar{\nu}}[\bar{\mu}, E \cap \alpha_\lambda]$, where $\bar{\nu} = \mathrm{Trans.\ Collapse}\ (M_\lambda) \cap \mathrm{ORD}$, and hence over any of its iterates, since iteration maps are Σ_1-elementary. But some such iterate is of the form $L_{\nu'}[\mu, E \cap \alpha_\lambda]$, and so we are done. ⊣

COROLLARY 1.4. *$R^{<s} = \bigcup\{R^t \mid t \subseteq s, t \neq s\}$ is $\leq\alpha$-distributive in $\hat{\mathscr{A}}_s$.*

PROOF. By Lemma 1.2 it suffices to prove distributivity in $\mathscr{A}_s^0$. But this is clear by induction on $|s|$, using Lemma 1.3 at successor stages. ⊣

LEMMA 1.5. *If $D \subseteq R^{<s}$, $D \in \hat{\mathscr{A}}_s$ is predense and $s \subseteq t \in S_{\alpha^+}$, then D is predense on R^t.*

PROOF. As in the proof of Corollary 1.4, it suffices to show that if $D \subseteq R^s$, $D \in \mathscr{A}_s$ is predense and $s \subseteq t \in S_{\alpha^+}$, then D is predense on R^t. Suppose $(u, \bar{u}) \in R^t$, and we will find an extension that meets D. We can assume that $s \in \mathrm{Rng}(\pi(\bar{u}))$ and that $D \in \mathrm{Rng}(\pi(\bar{u}) \restriction \mathscr{A}_{\bar{s}})$, where $\pi(\bar{u})(\bar{s}) = s$. Also assume that $\bar{s}$ does *not* lie on $(u)_1$ and that $|u| = |s'| + 1$ for some s', $\bar{s} < s' < s$.

Now let $(u', \bar{u}')$ be the least extension of $(u, \bar{u} \cap \mathscr{A}_{\bar{s}}) \in R^s$ meeting D. We are done if we show that $(u', \bar{u}) \leq (u, \bar{u})$, for then we can amalgamate $(u', \bar{u}')$ and $(u', \bar{u})$ to obtain an extension of $(u, \bar{u})$ meeting D. The worry is that some $t' * 1$ lies on $(u')_1$ but not on $(u)_1$, where $t' \in b_{t\restriction\eta}$ is being restrained by $(u, \bar{u})$ and $\eta \geq |s|$. But this is impossible if $\alpha(t') > \alpha(d(\bar{u}))$ as $[|u|, |u'|]$ lies strictly inside $(|s_0'|, |s_1'|)$ for $s_0' < s_1' < s$. So the only danger is that $t' * 1$ lies on $d(\bar{u})$. But $\bar{s}$ is an initial segment of $d(\bar{u})$ and by hypothesis $\bar{s}$ does not lie on $(u)_1$. So t' does not lie on $(u)_1$, which means that $t' * 1$ is not restrained after all. ⊣

REMARK. The proof of Lemma 1.5 is somewhat easier than with the usual coding due to the weaker type of restraint being used. (The last step of the proof takes advantage of the weaker restraint.)

This completes our discussion of the R^s forcing.

§2. The limit coding. In defining the limit coding we combine the R^s forcings of the previous section as in other coding theorems, with restrictions necessary for extendibility of conditions.

Fix a limit cardinal $\alpha < \kappa$ and $u \in S_\alpha$. We wish to define a forcing $\mathscr{P}^u$ for coding u below α.

We need as usual appropriate forms of $\square$ and $\diamondsuit$, which we now describe. Let $\underline{S} = \bigcup\{S_{\bar\alpha} \mid \bar\alpha \text{ a cardinal } < \kappa\}$.

Fact C. *There exists* $\langle C_s \mid s \in \underline{S}\rangle$ *with exactly the properties of* $\langle C_s \mid s \in S^+\rangle$ *in Fact* A.

Let $E = \{s \in \underline{S} \mid \text{o.t.}(C_s) = \omega\}$.

Fact D. *There exists* $\langle D_s \mid s \in E\rangle$ *such that* $D_s \subseteq H^0_s = H^{\mathscr{A}^0_s}_{\alpha(s)^+}$ *and:*

(a) $D \in \hat{\mathscr{A}}_s \neq \mathscr{A}^0_s, D \subseteq H^0_s \to \{\xi < |s| \mid s \restriction \xi \in E, D_{s\restriction\xi} = D \cap \xi\}$ *is stationary in* $\hat{\mathscr{A}}_s$.

(b) D_s *is uniformly* $\Delta_1\langle \mathscr{A}^0_s, C_s\rangle$ *for* $s \in E$.

(c) $s \in E$, Σ_1 *projectum* $(\hat{\mathscr{A}}_s) \neq \alpha(s) \to D_s = \phi$.

Note. Σ_1 projectum $(\hat{\mathscr{A}}_s)$, by definition, is the least ρ such that some $\Sigma_1(\hat{\mathscr{A}}_s)X \subseteq \rho$ is not an element of $\mathscr{A}_s$. (c) is needed to satisfy requirements in the proof of extendibility of conditions (see Lemma 2.3).

Now as a rough indication of the nature of $\mathscr{P}^u$ we first define $\mathscr{P}^u \subseteq \mathscr{A}_u$, a set of "quasiconditions" which will have to be thinned out in a number of ways to obtain the proper $\mathscr{P}^u$. An element of $\mathscr{P}^u - \mathscr{P}^{<u}$ (where $\mathscr{P}^{<u} = \bigcup\{\mathscr{P}^{u\restriction\xi} \mid \xi < |u|\}$) is a function p: Card $\cap\, \alpha \to V$ such that for $\beta \in \text{Dom}(p)$, $p(\beta) = (p_\beta, \bar p_\beta) \in R^{p_{\beta^+}}$, for limit $\beta \in \text{Dom}(p)$ we have (inductively) $p \restriction \beta \in \mathscr{P}^{p(\beta)}$ and $|p \restriction \beta|$ = least ξ such that $p \restriction \beta \in \mathscr{A}_{p_\beta \restriction \xi}$ is equal to $|p_\beta|$. We also insist that p *codes* u in the following sense: For $\xi < |u|$ and $\beta \in \text{Card} \cap \alpha$ define $M^\xi_\beta = \Sigma_1$ Skolem hull of $\beta \cup \{u \restriction \xi, \kappa\}$ in $\mathscr{A}_{u\restriction\xi}$ and $(b^\xi_\beta)^+$ = least $\phi^+_\delta < \phi^+_{\beta^+}$ such that $M^\xi_\beta \cap \beta^+ \leq \delta$ and o.t.$\{\delta' \mid \phi^+_{\delta'} < \phi^+_\delta\}$ is odd. Then we code u by: $u(\xi) = 1$ iff $b^\xi_{\beta^+} * 1$ lies on p^+_β for sufficiently large $\beta \in \text{Card} \cap \alpha$. Recall that if $\{\delta' \mid \phi^+_{\delta'} < \phi^+_\delta\}$ has odd ordertype then ϕ_δ is not an element of $b_{\phi_{\beta^+}}$ and so $\phi_\delta * 0$ and $\phi_\delta * 1$ are not restrained by the successor coding $R^{p_{\beta^+}}$.

To obtain $\mathscr{P}^u - \mathscr{P}^{<u}$ we impose a number of further requirements.

Requirements A (predensity reduction). Suppose $p \in \mathscr{P}^u - \mathscr{P}^{<u}$.

(A1) If $u \in E$ and $D_u \subseteq \mathscr{P}^{<u}$ is β-predense for all $\beta < \alpha$, then p meets D_u.

(A2) If $|u|$ is a successor, $D \subseteq \mathscr{P}^{<u}$ is predense and $D \in \mathscr{A}^0_u$ then p reduces D below some $\beta < \alpha$.

REMARK. D is β-predense if $\forall q \exists r (r \in D^*, r \restriction \beta = q \restriction \beta)$, where $D^* = \{r \mid r$ extends an element of $D\}$. And p reduces D below β if $\forall q \leq p \exists r \leq q (r \in D^*, (r)_\beta = r - r \restriction \beta$ equals $(q)_\beta)$.

Requirement B (restriction). If $p \leq q$ belong to $\mathscr{P}^u$ and $|q| \leq \xi < |p|$ then there exists $r \in \mathscr{P}^v - \mathscr{P}^{<v}$, $p \leq r \leq q$, where $v = u \restriction \xi$.

Requirement C (nonstationary restraint). Suppose $\mathscr{A}_u \models \alpha$ inaccessible. Then there exists a CUB $C \subseteq \alpha$ such that $C \in \mathscr{A}_u$ and $\beta \in C \to \bar p_\beta = \varnothing$.

The remaining Requirement D will be introduced at a later point when we discuss strong extendibility at successor stages.

Extendibility and distributivity for $\mathscr{P}^u$ are stated as follows. Let $q \leq_\beta p$ signify that $q \leq p$, $q \restriction \beta = p \restriction \beta$ and $(\mathscr{P})_\beta = \{(p)_\beta \mid p \in \mathscr{P}\}$.

$(*)_u$ $\qquad p \in \mathscr{P}^u, \beta \in \text{Card} \cap \alpha \to \exists q \leq_\beta p (q \in \mathscr{P}^u - \mathscr{P}^{<u})$.

$(**)_u$ $\qquad \forall \beta < \alpha ((\mathscr{P}^{<u})_\beta$ is $\leq \beta$-distributive in $\hat{\mathscr{A}}_u)$.

Also $\mathscr{A}^0_u \models \alpha$ inaccessible $\to \mathscr{P}^{<u}$ is Δ-distributive in $\hat{\mathscr{A}}_u$.

REMARK. Δ-distributivity asserts that if $\langle D_\beta \mid \beta \in \text{Card} \cap \alpha\rangle$ satisfies D_β is β^+-predense for all β then $\forall p \exists q \leq p$ (q meets each D_β).

$(*)_u$ and $(**)_u$ are proved via a simultaneous induction on $|u|$. The following consequences of predensity reduction are needed in the proof.

LEMMA 2.1 (chain condition for $\mathscr{P}^{<u}$). *Suppose* $(**)_u$ *holds. Then* $\mathscr{P}^{<u}$ *has the* α^+-*CC in* $\hat{\mathscr{A}}_u$.

PROOF. Suppose $D \subseteq \mathscr{P}^{<u}$ is predense and $D \in \hat{\mathscr{A}}_u$. Consider $D^* = \{p \in \mathscr{P}^{<u} \mid p$ reduces D below some $\beta < a\} \in \hat{\mathscr{A}}_u$. Then by $(**)_u$ and Lemma 1.2, D^* is β-predense for all $\beta < \alpha$. Apply Fact D to obtain $\xi < |u|$ such that $u \restriction \xi \in E$ and $D_{u\restriction\xi} = D^* \cap H^{\mathscr{A}^0_{u\restriction\xi}}_{\alpha^+}$. Thus by predensity reduction we have that $D^* \cap H^{\mathscr{A}^0_{u\restriction\xi}}_{\alpha^+} \in \mathscr{A}^0_u$ is predense on $\mathscr{P}^{<u}$, and therefore so is $D \cap H^{\mathscr{A}^0_{u\restriction\xi}}_{\alpha^+} \in \mathscr{A}^0_u$. $\dashv$

LEMMA 2.2 (persistence for $\mathscr{P}^{<u}$). *Suppose* $(**)_u$ *holds,* $D \subseteq \mathscr{P}^{<u}$ *is predense,* $D \in \hat{\mathscr{A}}_u$ *and* $u \subseteq v$. *Then* D *is predense on* $\mathscr{P}^v$.

PROOF. By restriction, if $p \in \mathscr{P}^v - \mathscr{P}^u$ then p extends some $q \in \mathscr{P}^u - \mathscr{P}^{<u}$. By the chain condition for $\mathscr{P}^{<u}$ we can assume that $D \in \mathscr{A}^0_u$ and hence by induction that $|u|$ is a successor. But then q reduces D below some $\beta < \alpha$ by predensity reduction, and hence so does p. So p is compatible with an element of D. $\dashv$

We can now turn to the proofs of $(*)_u$, $(**)_u$.

LEMMA 2.3. *Assume* $(**)_u$. *Then* $(*)_u$ *holds when* $|u|$ *is a limit ordinal.*

PROOF. We first claim that if $p \in \mathscr{P}^{<u}$ and $\langle D_\beta \mid \beta_0 \leq \beta < \alpha\rangle \in \mathscr{A}^0_u$ satisfy that $D_\beta \subseteq \mathscr{P}^{<u}$ is β^+-predense then $\exists q \leq_{\beta_0} p$ (q meets each D_β). We prove this by showing it with α replaced by $\beta_1 \leq \alpha$, by induction on β_1. The base case and successor case are clear, using $(**)_u$. If β_1 is singular then we can choose $\gamma_0 < \gamma_1 < \cdots$ approximating β_1, replace $\langle D_\beta \mid \beta_0 < \beta < \beta_1\rangle$ by $\langle E_i \mid \beta_0 \leq i < \beta_0^{(+j)},\ j =$ o.t.$(\gamma_i$'s$)\rangle$, where $E_i =$ all q meeting each D_β, $\beta_0 \leq \beta < \gamma_i$, and then we are done by induction. Finally, if β_1 is inaccessible we factor $\mathscr{P}^{<u}$ as $(\mathscr{P}^{<u})_{\beta_1^+} * \mathscr{P}^{(G)_{\beta_1^+}}$, where $(\mathscr{P}^{<u})_{\beta_1^+} = \{(q)_{\beta_1^+} \mid q \in \mathscr{P}^{<u}\}$, and first choose $(q)_{\beta_1^+} \leq (p)_{\beta_1^+}$ that reduces each D_β, $\beta_0 \leq \beta < \beta_1$, below β_1^+. This is possible using $(**)_u$ and the β_1^{++}-CC of $\mathscr{P}^{(G)_{\beta_1^+}}$, which we have by induction on α.

Now write $C_u = \{\nu^0_{u\restriction\xi_i} \mid i < \lambda\}$ and choose a successor cardinal $\beta_0 < \alpha$ to be at least as large as λ and the β in the statement of $(*)_u$, if $\lambda < \alpha$; otherwise let β_0 equal α. Now inductively define a subsequence $\langle \eta_j \mid j < \lambda_0\rangle$ of $\langle \xi_i \mid i < \lambda\rangle$ and $\langle p_j \mid j < \lambda_0\rangle$ as follows: $\eta_0 =$ least ξ_i such that the given $p \in \mathscr{P}^{<u\restriction\xi_i}$, $p_0 = p$; p_{j+1} is the least $q \leq_\beta p_j$ (if $\beta_0 < \alpha$; otherwise $q \leq_{\beta \cup \aleph_{i+1}} p_j$, where $\eta_j = \xi_i$) such that for all γ, $\beta_0 \leq \gamma < \alpha$, q meets all γ^+-predense $D \subseteq \mathscr{P}^{<u\restriction\eta_j}$, $D \in M^{\eta_j}_{\gamma^+} = \Sigma_1$ Skolem hull of $\gamma^+ \cup \{p, \kappa, \mu\}$ in $\langle \mathscr{A}^0_{u\restriction\eta_j}, C_{u\restriction\eta_j}\rangle$ (iff $\beta_0 < \alpha$; otherwise consider only γ between $\beta \cup \aleph_i$ and α, $\eta_j = \xi_i$) and $\eta_{j+1} =$ *least* ξ_i such that $p_{j+1} \in \mathscr{P}^{<u\restriction\xi_i}$; $p_\lambda =$ g.l.b.$\langle p_i \mid i < \lambda\rangle$ for limit $\lambda \leq \lambda_0$, $\eta_\lambda = \bigcup\{\eta_i \mid i < \lambda\}$ for limit $\lambda < \lambda_0$. The ordinal λ_0 is determined by the condition that $\eta_{\lambda_0} = \bigcup\{\eta_i \mid i < \lambda_0\}$ is equal to $|u|$.

We must verify that p_λ as defined above is indeed a condition for limit λ. (At successor stages, p_{j+1} is well-defined using Lemma 2.2.) For example we must show that for γ a cardinal $< \alpha$, p_{λ_γ} is μ, $A \cap \gamma$-reshaped. We need only consider $\gamma \geq \beta$, and in case $\beta_0 = \alpha$ we need only consider $\gamma \geq \beta \cup \aleph_i$ where $\eta_\lambda = \xi_i$. Now by construction if $\gamma \in M^{\eta_\lambda}_\gamma$ then p_{λ_γ} is $\pi[(\mathscr{P}^{<u\restriction\eta_\lambda})_\gamma]$-generic over $TC(M^{\eta_\lambda}_\gamma)$, where

$$\pi\colon M^{\eta_\lambda}_\gamma = (\Sigma_1 \text{ Skolem hull of } \gamma \cup \{p, \kappa, \mu\} \text{ in } \langle \mathscr{A}^0_{u\restriction\eta_\lambda}, C_{u\restriction\eta_\lambda}\rangle) \to TC(M^{\eta_\lambda}_\gamma)$$

is the transitive collapse. And $|p_{\lambda_\gamma}|$ is Σ_1-definably singularized over $TC(M^{\eta_\lambda}_\gamma)$ and

hence, by the first statement in (d) from Fact A (which applies here: see Fact C), $|p_{\lambda_\gamma}|$ is Σ_1-definably singularized over $\langle \mathscr{A}^0_{p_{\lambda_\gamma}}, C_{p_{\lambda_\gamma}}\rangle$. As $C_{p_{\lambda_\gamma}} \in L[\mu, A \cap \gamma, p^*_{\lambda_\gamma}]$, we have reshaping as desired. If $M^{\eta_\lambda}_\gamma \cap \alpha = \gamma$ then p_{λ_γ} is reshaped, as it must be the image of $u \restriction \eta_\lambda$ under the transitive collapse of $M^{\eta_\lambda}_\gamma$, in which case reshaping follows again from (d) in Fact A. Lastly if $\gamma' = \min(M^{\eta_\lambda}_\gamma \cap (\mathrm{ORD} - \gamma))$ then the above two arguments can be combined to first argue for the reshaping of $p_{\lambda_{\gamma'}}$; and then for that of p_{λ_γ}.

But we must also show that $p_\lambda \restriction \gamma \in \mathscr{A}_{p_{\lambda_\gamma}}$. As $p_\lambda \restriction \gamma$ is definable over $TC(M^{\eta_\lambda}_\gamma) \in L[\mu, A \cap \gamma, p_{\lambda\gamma}]$, this amounts to showing that $\nu_{p_{\lambda\gamma}}$ is large enough. Now as we have $(**)_{u\restriction\eta_\lambda}$, we know that (when $M^{\eta_\lambda}_\gamma \cap \alpha \neq \gamma$) $\mathscr{P}^{<u\restriction\eta_\lambda}$ has the α^+-CC in $\mathscr{A}_{u\restriction\eta_\lambda}$ by Lemma 2.1, and hence p_{λ_γ} is in fact $\tilde{\pi}'^{-1}[(\mathscr{P}^{<u\restriction\eta_\lambda})_\gamma]$-generic over $\tilde{\pi}'^{-1}[\mathscr{A}_{u\restriction\eta_\lambda}] = \hat{\mathscr{A}}$, where $\tilde{\pi}'$ is as in the second part of (d) in Fact A, π = inverse of the transitive collapse of $M^{\eta_\lambda}_\gamma$. We know that $\hat{C} = \tilde{\pi}'^{-1}[C_{u\restriction\eta_\lambda}]$ is definable over $\mathscr{A}' = \tilde{\pi}'^{-1}[\mathscr{A}_{u\restriction\eta_\lambda}]$ and $\nu' = \mathrm{ORD}(\mathscr{A}') <$ least p.r. closed ordinal greater than $\hat{\nu} = \mathrm{ORD}(\hat{\mathscr{A}})$. But $\hat{\mathscr{A}}[p_{\lambda_\gamma}] \models |p_{\lambda_\gamma}|$ is a cardinal. So $\hat{\mathscr{A}}_{p_{\lambda_\gamma}} = \hat{\mathscr{A}}[p_{\lambda_\gamma}]$ and $\nu' < \nu_{p_{\lambda_\gamma}}$, $\hat{C} \in \mathscr{A}_{p_{\lambda_\gamma}}$. If $M^{\eta_\lambda}_\gamma \cap \alpha = \gamma$ then there is no problem since $p_\lambda \restriction \gamma$ is definable over $\pi^{-1}[\langle \mathscr{A}^0_{u\restriction\eta_\lambda}, C_{u\restriction\eta_\lambda}\rangle]$, which iterates to $\langle \mathscr{A}^0_{p_{\lambda_\gamma}}, C_{p_{\lambda_\gamma}}\rangle$, π = inverse to the transitive collapse of $M^{\eta_\lambda}_\gamma$.

And we must verify Requirement (A1). (Requirements B and C for p_λ are easily checked.) If $M^{\eta_\lambda}_\gamma \cap \alpha \neq \gamma$ then $p_{\lambda_\gamma} \in E \to D_{p_{\lambda_\gamma}} = \phi$ (see (c) of Fact D), since (d) of Fact A implies that Σ_1 projectum $(\hat{\mathscr{A}}_{p_{\lambda_\gamma}}) = \tilde{\pi}'^{-1}(\Sigma_1 \text{ projectum } (\hat{\mathscr{A}}_{u\restriction\eta_\lambda})) > \alpha(p_{\lambda_\gamma}) = \gamma$. If $M^{\eta_\lambda}_\gamma \cap \alpha = \gamma$ and $u \restriction \eta_\lambda \notin E$ then $p_{\lambda_\gamma} \notin E$ since π preserves square sequences, π = inverse to the transitive collapse of $M^{\eta_\lambda}_\gamma$. Finally if $M^{\eta_\lambda}_\gamma \cap \alpha = \gamma$ and $u \restriction \eta_\lambda \in E$ then $D_{p_{\lambda_\gamma}} = \pi^{-1}[D_{u\restriction\eta_\lambda}]$ by virtue of (b) in Fact D, where π = inverse to the transitive collapse of $M^{\eta_\lambda}_\gamma$.

Finally set $q = p_{\lambda_0}$, and we have established $(*)_u$. $\dashv$

LEMMA 2.4. *Assume* $(*)_v$ *and* $(**)_v$ *for* $v \subseteq u$, $v \neq u$, *and also assume* $|u|$ *limit. Then* $(**)_u$ *holds.*

PROOF. We may assume that $\hat{\mathscr{A}}_u \neq \mathscr{A}^0_u$. This requires only a small change in the construction of Lemma 2.3. Instead of extending along C_u as in that proof we extend along a closed subsequence $C \subseteq C_{u\restriction\xi}$ for some $\xi < |u|$ with the property that if $\langle D_i \mid i < \beta\rangle$ is the given sequence of predense subsets of $(\mathscr{P}^{<u})_\beta$ then o.t.$(C) = \beta$ and $q \in \mathscr{P}^{u\restriction\xi_i} \to \exists r \leq q$ ($r \in \mathscr{P}^{u\restriction\xi_{i+1}}$, r meets D_i) where $C = \{\xi_i \mid i < \beta\}$. Moreover $u \restriction \xi_\lambda \notin E$ and $C \cap \xi_\lambda \in \mathscr{A}_{u\restriction\xi_\lambda}$ for limit λ. It is easy to obtain C by choosing $\xi < |u|$ so that $\mathrm{cof}(\xi) \geq \beta$, $D_i \cap (\mathscr{P}^{<u\restriction\xi})_\beta$ predense on $(\mathscr{P}^{<u\restriction\xi})_\beta$ for all $i < \beta$ and then taking an appropriate subsequence of $C_{u\restriction\xi}$. The Δ-distributivity argument is similar. $\dashv$

We now turn to the case of $|u|$ successor, where an additional requirement must be imposed for the sake of proving $(**)_u$.

LEMMA 2.5. *Suppose* $(**)_u$ *holds and* $|u|$ *is a successor ordinal. Then* $(*)_u$ *holds.*

PROOF. We can assume that $p \in \mathscr{A}_v - \mathscr{A}^0_v$, where $v = u \restriction (|u| - 1)$. Also note that $C_u = \langle \xi_j \mid j < \lambda\rangle$ has ordertype ω. Now proceed as in the proof of Lemma 2.3, making successive extensions $p \geq_\beta p_0 \geq_\beta p_1 \geq \cdots$ so that p_{j+1} meets all γ^+-predense $D \subseteq \mathscr{P}^{<u} \cap N^{\xi_j}_{\gamma^+}$, where $N^{\xi_j}_{\gamma^+} = \Sigma_1$ Skolem hull of $\gamma^+ \cup \{u, \kappa\}$ in $\mathscr{A}_u \restriction \xi_j$, for all $\gamma \in [\beta, \alpha)$. If we set $\hat{q} = \text{g.l.b.}\langle p_i \mid i < \omega\rangle$ then $\hat{q}$ meets the necessary requirements for being a condition at all γ with the exception of γ such that $\gamma \in C = \{\gamma \mid M_\gamma \cap \alpha = \gamma\}$ or $\gamma = \alpha$, where $M_\gamma = \bigcup\{N^{\xi_j}_\gamma \mid j < \omega\} = \Sigma_1$ Skolem hull of $\gamma \cup \{u, \kappa\}$ in $\mathscr{A}_u$. The reason is that for $\gamma \in \alpha - C$, T_γ = transitive collapse (M_γ) belongs to $\mathscr{A}^1_{\hat{q}_\gamma}$, since $T_\gamma \models |\hat{q}_\gamma|$ is a cardinal and $\hat{q}_\gamma$ is generic over T_γ.

To extend $\hat{q}$ to a condition $q \in \mathscr{P}^u$ we must do two things. First extend $\hat{q}(\gamma^+)$ for large enough $\gamma \geq \beta$ so as to code $u(|v|) = 0$ or 1. This is easily done as there are no conflicts between the successor and limit codings. Secondly for $\gamma \in C$ we must define $q(\gamma) = (q_\gamma, \bar{q}_\gamma)$ where $\bar{q}_\gamma = \bar{\hat{q}}_\gamma$ and $q_\gamma = \hat{q}_\gamma * u(|v|)$. The only worry left is whether the restraint $\bar{\hat{q}}_\gamma$ will allow us to do this.

But notice that by definition of $p \in \mathscr{P}^u$, $u(|v|) = 1 \to |v|$ is not restrained by p on a CUB set $D \subseteq \alpha$. We know that $C - \beta'$ is contained in D for some $\beta' < \alpha$, and so we can obtain the desired q by only coding $u(|v|)$ at $q(\gamma^+)$ for $\gamma \geq \beta'$. $\dashv$

REMARK. Lemma 2.5 and Requirement C together guarantee that if G is $\mathscr{P}^s$-generic, $s \in S_{\alpha^+}$, then $G \restriction \alpha$ codes not only G_α but s and $A \cap \alpha^+$ as well. In other words, the restraint imposed by $G \restriction \alpha$ does not interfere with G_α's ability to code s, $A \cap \alpha^+$.

Finally we turn to the successor case of $(**)_u$.

LEMMA 2.6. *Suppose* $(*)_u$ *and* $(**)_v$ *hold for* $v \subseteq u \neq v$. *Then* $(**)_u$ *holds.*

PROOF. We must show that if $v = u \restriction (|u| - 1)$ and $p \in (\mathscr{P}^v)_\beta - (\mathscr{P}^{<v})_\beta$, $\langle D_i \mid i < \beta \rangle \in \mathscr{A}_v$ are predense on $(\mathscr{P}^v)_\beta$ then there exists $q \leq p$, q meets each D_i. For simplicity assume that $\beta = \omega$. Our argument will be exactly as in our proof of the ordinary coding theorem (over L) with the sole exception of the use of iteration methods in the proof of strong extendibility.

DEFINITION. Suppose $f(\beta) = M_\beta$ is a function in $\mathscr{A}_v$ from $\mathrm{Card}^+ \cap \alpha$ (=all successor cardinals $< \alpha$) into $\mathscr{A}_v$ such that $\mathrm{card}(M_\beta) \leq \beta$ for all $\beta \in \mathrm{Dom}(f)$ and $p \in \mathscr{P}^v$. Then $\Sigma_f^p = \{q \in \mathscr{P}^v \mid \forall \beta \in \mathrm{Dom}(f)(q(\beta)$ meets all predense $D \subseteq R^{p_\beta +}$, $D \in M_\beta)\}$.

SUBLEMMA 2.7. Σ_f^p *is dense below* p *in* $\mathscr{P}^v$.

Before proving Sublemma 2.7 we establish the lemma, assuming it. Choose a limit ordinal $\lambda < v_v$ such that $p, \langle D_i \mid i < \omega \rangle \in \mathscr{A}_v \restriction \lambda$ and so that Σ_1 cofinality $(\mathscr{A}_v \restriction \lambda) = \omega$. Choose $\lambda_0 < \lambda_1 < \cdots$ cofinal below λ such that $\langle \lambda_i \mid i < \omega \rangle \in \Sigma_1(\mathscr{A}_v \restriction \lambda)$ in parameter x and $p, \langle D_i \mid i < \omega \rangle$, $x \in \mathscr{A}_v \restriction \lambda_0$. Define: $M_\gamma^i =$ least $M <_{\Sigma_1} \mathscr{A}_v \restriction \lambda_i$ such that $\gamma \cup \{p, x, \langle D_i \mid i < \omega \rangle, \kappa\} \subseteq M$ for each $\gamma \in \mathrm{Card}^+ \cap \alpha$. Define $f_i(\gamma) = M_\gamma^i$.

Choose $p = p_0 \geq p_1 \geq \cdots$ successively so that p_{i+1} meets D_i and $\Sigma_{f_i}^{p_i}$. Set $p^* =$ g.l.b.$\langle p_i \mid i < \omega \rangle$. We must show that p^* is a well-defined condition. Thanks to $(**)_v$ it will suffice to show that if $D \in M_\gamma^i \cap \mathscr{A}_v^0$ is predense on $(\mathscr{P}^{<v})_\gamma$, $\gamma \in \mathrm{Card} \cap \alpha$, then some p_j reduces D below γ. (For then, p_γ^* is generic over the transitive collapse of $M_\gamma^i \cap \mathscr{A}_v^0$.)

Choose $j \geq i$ so that p_k reduces D no further than p_j for $k > j$. Let γ' be least so that p_j reduces D below γ'. Then $\gamma' < \alpha$ by predensity reduction for p. If $\gamma' \leq \gamma$ then of course p_j reduces D below γ and we are done. If $\gamma' > \gamma$ is a double successor cardinal then we reach a contradiction, since by definition p_{j+1} will reduce D further. If $\gamma' > \gamma$ is the successor to a limit cardinal δ then notice that $D^{(p_j)_{\gamma'}} \cap \mathscr{A}_{p_{k_\delta}}^0$ belongs to $\mathscr{A}_{p_{k_\delta}}$ and is predense on $(\mathscr{P}^{<p_k(\delta)})_\gamma$ for some k, since

$$\bigcap \{|p_{k_\delta}| \mid k < \omega\} = \delta^+ \cap \bigcup \{M_\delta^k \mid k < \omega\}.$$

So by predensity reduction for p_k at δ, D is reduced below some $\delta' < \delta$, a contradiction. Lastly if γ' is a limit cardinal then the preceding argument applies, replacing γ' by $(\gamma')^+$.

PROOF OF SUBLEMMA 2.7. First suppose that α is inaccessible in $\mathscr{A}_v$. We want to extend p to meet Σ_f^p. We can assume that $p \in \mathscr{A}_v - \mathscr{A}_v^0$ (by persistence) and choose a

limit ordinal $\nu < \nu_v$ such that $p, f \in \mathscr{A}_v \restriction \nu$. Let $C = \{\beta < \alpha \mid \beta = \alpha \cap \Sigma_1$ Skolem hull of $\beta \cup \{p, f, \kappa\}$ in $\mathscr{A}_v \restriction \nu\}$, a CUB subset of α. Enumerate $C = \{\beta_i \mid i < \alpha\}$ and proceed as follows: $p_0 =$ least $q \leq p$ such that q meets $\Sigma^p_{f \restriction \beta_0}$ and $p_{i+1} =$ least $q \leq_{\beta_i} p_i$ such that q meets $\Sigma^p_f \restriction \beta_i$; for limit $\lambda \leq \alpha$, $p_\lambda =$ g.l.b.$\langle p_i \mid i < \lambda\rangle$. Then p_{i+1} is well-defined when p_i is, since $\mathrm{Dom}(f) \subseteq \mathrm{card}^+$ and we can use induction on α to get the density of $\Sigma^p_{f \restriction \beta_i}$. For limit $\lambda \leq \alpha$ notice that $p_\lambda \restriction \beta_\lambda \in \mathscr{A}_{p_{\lambda_{\beta_\lambda}}} = \mathscr{A}_{p_{\beta_\lambda}}$ since $\beta \in C \to f \restriction \beta, C \cap \beta \in \mathscr{A}_{p_\beta}$. So p_λ is well-defined and we can let $q = p_\alpha \in \Sigma^p_f$, $q \leq p$.

In the singular case we can repeat the above argument, provided we have the following.

STRONG EXTENDIBILITY. *Suppose $g \in \mathscr{A}_v, g(\beta) \in H_{\beta^+}$ for all $\beta \in \mathrm{Card} \cap (\beta_0, \alpha)$ and $p \in \mathscr{P}^v$. Then there is $q \leq_{\beta_0} p$ such that $g \restriction \beta \in \mathscr{A}_{q_\beta}$ for all $\beta \in \mathrm{Card} \cap (\beta_0, \alpha]$.*

For, strong extendibility allows us to choose $q \leq p$, a CUB $C \subseteq \alpha$ and $\nu < \nu_v$ such that for $\beta \in \mathrm{Lim}(C)$, $T^\nu_\beta =$ transitive collapse (M^ν_β) and $C \cap \beta$ belong to $\mathscr{A}_{q_\beta}$, where $M^\nu_\beta = \Sigma_1$ Skolem hull of $\beta \cup \{p, f, \kappa\}$ in $\mathscr{A}_v \restriction \nu$. Then $\beta \in \mathrm{Lim}(C) \to C \cap \beta$ and $f \restriction \beta \in \mathscr{A}_{q_\beta}$, so our preceding argument applies.

We now break down strong extendibility into the ramified form in which it will be proved. For any $\nu^0_v \leq \nu < \nu_v$ and $k < \omega$ let $B^{\nu,k}$ denote the Σ_k master code structure for $\hat{\mathscr{A}}_v \restriction \nu$, where $\mathscr{A}_v = \langle \mathscr{A}_v, C_v\rangle$. By this we mean the following. We know that $\hat{\mathscr{A}}_v \restriction \nu$ is Σ_1-projectible so we let $B^{\nu,0} = \mathrm{Core}(\hat{\mathscr{A}}_v \restriction \nu) =$ transitive collapse of Σ_1 Skolem hull of $\rho^\nu_1 \cup \{\kappa, p\}$ in $\hat{\mathscr{A}}_v \restriction \nu$, where $\rho^\nu_1 = \Sigma_1$ projectum of $\hat{\mathscr{A}}_v \restriction \nu$ and $p =$ least parameter witnessing this fact. Note that $\alpha \leq \rho^\nu_1 \leq \kappa$. Then $B^{\nu,1} = \Sigma_1$ master code structure for $B^{\nu,0}$. In general $B^{\nu,k} = \Sigma_k$ master code structure for $B^{\nu,0}$, in the usual sense. Note that $B^{\nu,0}$ is k-sound for all k.

Let $M^{\nu,k}_\beta = \Sigma_1$ Skolem hull of $\beta \cup \{\kappa(B^{\nu,k-1}), p(B^{\nu,k-1})\}$ in $B^{\nu,k-1}$ for $\beta < \alpha$ and $k > 0$ (where $p(B^{\nu,k-1}) =$ standard parameter for $B^{\nu,k-1}$, $\kappa(B^{\nu,k-1}) =$ measurable of $B^{\nu,k-1}$ if $k = 1$). We use TC to denote "transitive collapse".

SE(v, k). Suppose $p \in \mathscr{P}^v$ and $\beta_0 < \alpha$. Then there exists $q \leq_{\beta_0} p$ such that $TC(M^{\nu,k}_\beta) \in \mathscr{A}_{q_\beta}$ for all $\beta > \beta_0$.

It is clear that strong extendibility is equivalent to the conjunction of $\mathbf{SE}(\nu, k)$ for $\nu < \nu_v, 0 < k < \omega$. $SE(\nu, k)$ is proved by induction on ν, and for fixed ν by induction on k. However to succeed with this induction we must impose one further requirement on our conditions.

Requirement D. Suppose $p \in \mathscr{P}^v - \mathscr{P}^{<v}$ and let ν, k be least so that p is $\Sigma_{k+1}(\hat{\mathscr{A}}_v \restriction \nu)$. Then $TC(M^{\nu,k}_\beta) \in \mathscr{A}_{p_\beta}$ for sufficiently large $\beta \in \mathrm{Card} \cap \alpha$.

The proof of $(*)_v$ shows that Requirement D is met in that construction and therefore $SE(\nu^0_v, 1)$ does hold, the base case of our induction. Note that $SE(\nu, k)$ is automatic by induction unless Σ_k projectum $(\hat{\mathscr{A}}_v \restriction \nu) = \alpha$. Suppose $k = 1$. If ν is a limit ordinal then so is $\bar{\nu} = \mathrm{ORD}(B^{\nu,0})$ and we can use a $\Sigma_1(B^{\nu,0})$-approximation to $\bar{\nu}$ and induction to obtain $q \leq p$ obeying $SE(\nu, 1)$ (using the Σ_f's, $f \in \hat{\mathscr{A}}_v \restriction \nu$). Similarly if ν is a successor use $\langle \Sigma_k(B^{\nu,0} \restriction \bar{\nu}') \mid k < \omega\rangle$ to approximate $\Sigma_1(B^{\nu,0})$ where $\bar{\nu} = \bar{\nu}' + 1$ (using also the Σ_f's, f definable over $\beta^{\nu,0} \restriction \bar{\nu}'$).

Suppose $k > 1$. By induction we can assume that $TC(M^{\nu,k-1}_\beta) \in \mathscr{A}_{p_\beta}$ for large enough β and by Requirement D we can assume that p is $\Sigma_k(\hat{\mathscr{A}}_v \restriction \nu) = \Sigma_k(B^{\nu,0})$. If $C = \{\beta < \alpha \mid \beta = \alpha \cap M^{\nu,k}_\beta\}$ is unbounded in α then successively extend $p \restriction \beta$ for $\beta \in C$ as in the inaccessible case so that $TC(M^{\nu,k}_\beta) \in \mathscr{A}_{q_{\beta'}}$ for $\beta' < \beta$. There is no problem at limits since $TC(M^{\nu,k}_\beta)$, $C \cap \beta \in \mathscr{A}_{p_\beta}$ for $\beta \in C$ ($B^{\nu,0}$ iterates to $\hat{\mathscr{A}}_v \restriction \nu$ and the iteration preserves Σ_k definability).

If α is $\Sigma_k(\hat{\mathscr{A}_v} \restriction v)$-singular then choose a continuous cofinal $\Sigma_k(\hat{\mathscr{A}_v} \restriction v)$ sequence $\beta_0 < \beta_1 < \cdots$ below α of ordertype $\lambda_0 = \text{cof}(\alpha)$. Also choose β_{i+1} large enough so that $M^{v,k-1}_{\beta_{i+1}} \models \beta_i$ is defined. This is possible since $\Sigma_k(\hat{\mathscr{A}_v} \restriction v) = \Sigma_1(B^{v,k-1})$ for subsets of α and $B^{v,k-1} = \bigcup\{M^{v,k-1}_\beta \mid \beta < \alpha\}$ unless Σ_{k-1} projectum $(\hat{\mathscr{A}_v} \restriction v) > \alpha$, in which case we can approximate $B^{v,k-1}$ as in the case $k = 1$. Now define N^i_β for $i < \lambda_0$, $\beta < \beta_i$ as follows: $N^i_\beta = \Sigma_1$ Skolem hull of $\beta \cup \{p(B^{v,k-1})\}$ in $M^{v,k-1}_{\beta_i}$. Then $\langle TC(N^i_\beta) \mid \beta < \beta_i\rangle \in \mathscr{A}_{p_{\beta_i}}$ for $i < \lambda_0$ since it is easily defined from $M^{v,k-1}_{\beta_i} \in \mathscr{A}_{p_{\beta_i}}$. Successively λ_0-extend $p \restriction \beta_i$, producing $p = p_0 \geq_{\lambda_0} p_1 \geq_{\lambda_0} \cdots$ where $TC(N^i_\beta) \in \mathscr{A}_{p_{i_\beta}}$ for $\beta \in (\lambda_0, \beta_i)$. This is possible by induction on α and since $TC(N^i_\beta)$ is easily defined from $\langle TC(N^i_{\beta'}) \mid \beta' < \beta\rangle$ for limit $\beta < \beta_i$. We must also require that $p_{i+1} \geq_{\lambda_0} p_i$ meets $\Sigma^{p_i}_{f_i}$ when $f_i(\beta) = N^i_\beta$. Then p_λ is well-defined for limit $\lambda \leq \lambda_0$ and $\mathscr{A}_{p_{\lambda_{o_\beta}}}$ contains $\langle TC(N^i_\beta) \mid i < \lambda_0\rangle$ and hence $TC(M^{v,k}_\beta)$ for $\beta > \lambda_0$. Finally use induction to fill in on $(0, \lambda_0]$ so that $SE(v,k)$ is satisfied.

Finally there is the intermediate case where α is $\Sigma_k(\hat{\mathscr{A}_v} \restriction v)$-regular but $C = \{\beta < \alpha \mid \beta = \alpha \cap M^{v,k}_\beta\}$ is bounded in α. Then $\Sigma_{k+1}(\hat{\mathscr{A}_v} \restriction v)$-$\text{cof}(\alpha) = \omega$ and we apply induction to produce $p = p_0 \geq p_1 \geq \cdots$, so that $p_{i+1} \restriction [\beta_i, \beta_{i+1}]$ obeys $SE(v,k)$ where $\beta_0 < \beta_1 < \cdots$ is a cofinal ω-sequence of successor cardinals below α. Let $q = \text{g.l.b.}\langle p_i \mid i < \omega\rangle$.

This completes the proof of Sublemma 2.7 and hence of $(**)_u$. $\dashv$

§3. Proof of the theorem. As remarked earlier we can assume that $V = L[\hat{A}]$, where $H_\kappa = L_\kappa[\hat{A}]$ for all infinite cardinals κ (in particular, GCH holds). By standard L-coding techniques we can then code $\hat{A}$ so as to produce $L[A, \mu]$, where $\hat{A} \cap \kappa = A \cap \kappa$, $A \subseteq \kappa^+$, no subsets of κ are added and A is L-reshaped: for any $\xi < \kappa^+$, $L[A \cap \xi] \models \text{Card}(\xi) \leq \kappa$. Note that as no subsets of κ are added, μ is still a measure in $L[\mu, A]$.

Now we must use the $\mathscr{P}^u$ forcings, $u \in S_\kappa$, from §2 to code A by a real in such a way as to allow for measure preservation. We first define large structures $\mathscr{A}_\xi$, $\xi < \kappa^+$, for controlling the A-coding. There are in fact two codings taking place simultaneously at κ, one for coding a $j(\mu)$-reshaped $B \subseteq \kappa^+$ and the other for coding the L-reshaped A. They are related by: $\eta \in A \leftrightarrow \langle 3, \eta_\xi\rangle \in B$, where $\eta_\xi = \text{ORD}(\mathscr{A}_\xi)$.

DEFINITION. The ordinals η_ξ and η^0_ξ are defined by induction on $\xi \in [\kappa, \kappa^+)$ as follows:

$\eta^0_\kappa = \kappa$, and $\eta^0_\xi = \bigcup\{\eta_{\xi'} \mid \xi' < \xi\}$ otherwise.

η_ξ = least p.r. closed $\eta > \eta^0_\xi$ such that $\mathscr{A}_\xi = \langle L_\eta[\mathscr{A}^0_\xi], \mu \restriction L_\eta[\mathscr{A}^0_\xi]\rangle$ is amenable, $\mathscr{A}_\xi \models \text{Card}(\xi) \leq \kappa$ and in addition $\mathscr{A}_\xi$ is closed under: $< \in \mathscr{A}_\xi$ a well-ordering of $\kappa \to \text{o.t.}(<) \in \mathscr{A}_\xi$.

Note that η_ξ is well-defined as $\mu \restriction L[A \cap \xi, \mu]$ is a normal measure in $L[A \cap \xi, \mu]$. We let $\mathscr{A}^0_\kappa = \langle L_\kappa[A], A \cap \kappa\rangle$ and $\mathscr{A}^0_{\xi+1} = \langle \mathscr{A}_\xi, (A \cap \xi)^*\rangle$, where $(A \cap \xi)^* = \{\eta_{\xi'} \mid \xi' \in A \cap \xi\}$ and $\mathscr{A}^0_\xi = \bigcup\{\mathscr{A}^0_{\xi'} \mid \xi' < \xi\}$ for limit ξ greater than κ. Note that each $\langle \mathscr{A}^0_\xi, \mu \restriction \mathscr{A}^0_\xi\rangle$ is amenable and, for $\xi > \kappa$, $\eta^0_\xi = (\kappa^+)^{M_\xi}$ when M_ξ is the ultrapower of $\mathscr{A}^0_\xi$ via $\mu \restriction \mathscr{A}^0_\xi$.

DEFINITION. S_κ = all u: $[\kappa, |u|) \to 2$, $|u| < \kappa^+$ which are $A \cap \kappa$, $j(\mu)$-reshaped: for $\xi \leq |u|$, $L[A \cap \kappa, j(\mu), u \restriction \xi] \models \text{Card}(\xi) \leq \kappa$. Here we are using $j(\mu)$ to denote the image of μ under the embedding j: $V \to M$ determined by the measure μ. Note that $j(\mu)$ is a measure in $M \supseteq L[A \cap \kappa, j(\mu), u \restriction \xi]$ for $\xi \leq |u|$.

For $u \in S_\kappa$ we can define the forcing $\mathscr{P}^u$ exactly as in §2, except with κ, μ replaced

by $j(\kappa)$, $j(\mu)$. However we restrict ourselves here to only those $u \in S_\kappa$ which are consistent with Requirement A of coding.

DEFINITION. $u \in S_\kappa$ is *OK* if $u \restriction \eta \in \mathscr{A}_\xi$ whenever $\eta < \eta_\xi$ and for $\langle 3, \eta_\xi \rangle \in \mathrm{Dom}(u)$, $u(\langle 3, \eta_\xi \rangle) = A(\xi) = 0$ or 1.

The desired forcing $\mathscr{P}$ consists of all conditions $p = \langle p \restriction \kappa, u(p), \bar{u}(p) \rangle$ satisfying the following:

(a) $p \restriction \kappa \in \mathscr{P}^{u(p)}$, $|p \restriction \kappa| = |u(p)|$ and $u(p) \in S_\kappa$ is OK.

(b) $(u(p), \bar{u}(p))$ is a condition in R^v where $v = j(p)_{\kappa^+}$.

And $p \leq q$ in $\mathscr{P}$ iff $p \restriction \kappa \leq q \restriction \kappa$ in $\mathscr{P}^{u(p)}$, $(u(p), \bar{u}(p)) \leq (u(q), \bar{u}(q))$ in R^v, $v = j(p)_{\kappa^+}$.

Extendibility for $\mathscr{P}$ follows from extendibility for $\mathscr{P}^u$, $u \in S_\kappa$, given the following claim.

Claim. Suppose $p \in \mathscr{P}$ and $\delta < \kappa^+$. Then there exists $q \leq p$, $\|q\| \geq \delta$, where $\|q\| =$ least ξ: $q \restriction \kappa \in \mathscr{A}_\xi$.

As the claim is proved using distributivity methods, we treat distributivity first (assuming the claim) and prove the claim later.

Distributivity. For $\alpha < \kappa$, $\mathscr{P}_\alpha = \{\langle p \restriction [\alpha, \kappa), u(p), \bar{u}(p) \rangle \mid p \in \mathscr{P}\}$ is $\leq \alpha$-distributive. Also, $\mathscr{P}$ is Δ-distributive.

This is established as follows, using the distributivity properties of the $\mathscr{P}^u$ forcings, $u \in S_\kappa$. Suppose $\alpha < \kappa$ and $\langle \Delta_i \mid i < \alpha \rangle$ are predense on $\mathscr{P}_\alpha$ and $p \in \mathscr{P}_\alpha$. We can make successive extensions $p = p_0 \geq p_1 \geq \cdots$ of p in α steps so that at limit stages $\lambda \leq \alpha$, $\|p_\lambda\| = \eta_\lambda = \kappa^+ \cap N_\lambda$ for some $N_\lambda \prec L_{\kappa^{++}}[A, \mu]$ such that A, μ, p, $\langle \Delta_i \mid i < \alpha \rangle \in N_\lambda$. To verify that p_λ is a well-defined condition we must arrange that $u(p_\lambda) = \bigcup \{u(p_i) \mid i < \lambda\}$ is $A \cap \kappa$, $j(\mu)$-reshaped and $p_\lambda \restriction \kappa$, $d(\bar{u}(p_\lambda)) \in \mathscr{A}_{u(p_\lambda)}$.

Let $j_\lambda \colon \mathscr{A}^0_{\eta_\lambda} \to M_\lambda$ be the ultrapower and $u^*_\lambda = \bigcup \{j_\lambda(u(p_i)) \mid i < \lambda\} = \bigcup j_\lambda[u(p_\lambda)]$. We claim that $u(p_\lambda)$ generically codes u^*_λ via the forcing $\mathscr{P}^*_\kappa$, where $\mathscr{P}^* = \bigcup \{j_\lambda[\mathscr{P}^{u(p_i)}] \mid i < \lambda\}$. Indeed, Δ-distributivity of $\mathscr{P}^{u(p_\lambda)}$ implies that any predense $\Delta^* \subseteq \mathscr{P}^*_\kappa$, $\Delta^* \in L_{\mathrm{ORD}(M_\lambda)}[j_\lambda[A \cap \kappa], u^*_\lambda]$ is reduced by some $j_\lambda(p_i)$ below κ^+, and therefore v is $\mathscr{P}^*_{\kappa^+_\lambda}$-generic where $v = \bigcup \{j_\lambda(p_i)_{\kappa^+_\lambda} \mid i < \lambda\}$ and κ^+_λ denotes $(\kappa^+)^{M_\lambda} = \eta_\lambda$. But we can easily arrange in the construction of the p_i's that for limit λ, $u(p_\lambda)$ is R^v-generic (v defined as above) by choosing $(u(p_{i+1}), \bar{u}(p_{i+1}))$ appropriately. Thus $u(p_\lambda)$ is $\mathscr{P}^*_\kappa$-generic and hence codes u^*_λ.

Finally we see that reshaping of $u(p_\lambda)$ (and the properties $p_\lambda \restriction \kappa$, $d(\bar{u}(p_\lambda)) \in \mathscr{A}_{u(p_\lambda)}$) is guaranteed provided we can show that $A^* = \bigcup \{j_\lambda(A \cap \eta_i) \mid i < \lambda\}$ is coded by u^*_λ, $j_\lambda(\mu \restriction \mathscr{A}^0_\lambda)$. Equivalently, we must show that $A \cap \eta_\lambda$ is coded by $u(p_\lambda)$, $\mu \restriction \mathscr{A}^0_\lambda$. But $\xi \in A \cap \eta_\lambda$ iff $u(p_\lambda)(\langle 3, \eta_\xi \rangle) = 1$ and $\langle \eta_\xi \mid \xi < \lambda \rangle$ can be inductively recovered from $u(p_\lambda)$, $\mu \restriction \mathscr{A}^0_\lambda$.

This demonstrates the reshaping of $u(p_\lambda)$. The reshaping of p_{λ_β} for $\beta \in [\alpha, \kappa)$ follows similarly, using the methods of §2. Finally, Δ-distributivity is a straightforward modification of the above.

We are left with our claim concerning extendibility. To prove it, we use:

Fact F. *There exists $\langle C_\xi \mid \xi < \kappa^+ \rangle$ such that $C_\xi \subseteq \eta^0_\xi$ is CUB,* o.t.$(C_\xi) \leq \kappa$ *and $\eta \in \mathrm{Lim}(C_\xi) \to \eta = \eta_{\xi'}$, where $C_{\xi'} = C_\xi \cap \eta$. Moreover C_ξ is uniformly definable as an element of $\mathscr{A}_\xi$.*

Proof of the Claim. Assume the claim for $\delta' < \delta$. Successively extend p to $p = p_0 \geq p_1 \geq \cdots$, where $\|p_i\| \in C_\delta$. By induction we can assume that δ is a limit ordinal and that p_{i+1} can be chosen to properly extend p_i; we must guarantee that $p_\lambda =$ g.l.b.$\langle p_i \mid i < \lambda \rangle$ is a well-defined condition for limit λ. To do so, arrange as in

distributivity that $u(p_\lambda)$ generically codes $u^*_\lambda = \bigcup\{j_\lambda(u(p_i)) \mid i < \lambda\}$. This guarantees the reshaping of $u(p_\lambda)$ and of p_{λ_β} for $\beta < \kappa$. $\dashv$

We have shown that $\mathscr{P}$ preserves cofinalities and that if G is $\mathscr{P}$-generic then $V[G] = L[\mu, R]$ for some real R. Finally we show:

THEOREM. *If G is $\mathscr{P}$-generic then κ is measurable in $V[G]$ via a measure μ^* extending μ.*

PROOF. Let μ arise from j: $V \to M$. We must select a $G^* \supseteq j[G]$ such that G^* is $j(\mathscr{P})$-generic over M, for then j extends to j^*: $V[G] \to M[G^*]$. Consider $\hat{G}^* = \{p^* \in j(\mathscr{P}) \mid$ for some $p \in G$, p^* and $j(p)$ agree at all $\alpha < j(\kappa)$ except $p^*(\kappa) = (u(p), \bar{u}(p))\}$. Then $\hat{G}^*$ is compatible by the $\mathscr{P}$-genericity of G. If $\Delta^* \subseteq j(\mathscr{P})$ is predense, $\Delta^* \in M$, then by Δ-distributivity of $j(\mathscr{P})$, $\hat{G}^*$ reduces Δ^* below κ^+. But then $\hat{G}^*$ meets Δ^* by the $\mathscr{P}$-genericity of G. So $G^* = \{p^* \mid p^*$ is extended by some element of $\hat{G}^*\}$ is the desired $j(\mathscr{P})$-generic. $\dashv$

§4. Extensions and applications.

THEOREM 4.1. *Suppose $L[\mu] \models \mu$ is a measure and $0^\dagger$ exists. Then there exists a real R, $0^\dagger \notin L[\mu, R]$, which is not set-generic over $L[\mu]$ such that $L[\mu, R] \models \mu$ extends to a measure. Moreover, R can be chosen independently of μ.*

PROOF. Let $L[\mu_0] \models \mu_0$ is a measure on κ_0, where κ_0 is the least ordinal measurable in an inner model. Then κ_0 is countable and there is a class of indiscernibles for $L[\mu_0]$ (i.e., $\mu_0^\#$ exists). So using the technique of the proof of Theorem 0.2 of Beller, Jensen and Welch [82], we can produce $R \subseteq \omega$ so that $L[\mu_0, R] \models \mu_0$ extends to a measure μ_0^* and R generically codes the ϕ-class over $L[\mu_0]$. Now if we iterate μ_0^* to μ^* then μ^* extends μ and R is generic over $L[\mu]$. So $L[\mu, R] \models \mu$ extends to a measure, $0^\dagger \notin L[\mu, R]$ and R is not set-generic over $L[\mu]$. $\dashv$

THEOREM 4.2 (Coding over K). *Suppose $\langle V, A\rangle \models ZFC + GCH$. Then there is a cardinal and cofinality preserving forcing for producing a real R such that $V[R] \models V = K^R$, A is definable with parameter R and every cardinal which is Ramsey in V is still Ramsey.*

PROOF. We use the coding of §2 as long as #'s exist: As we do not necessarily have the measure μ at our disposal, we work with mice instead. For example, if every subset of α has a sharp then $s \in S_\alpha$ must be $K^{A \cap \alpha}$-reshaped, meaning that $\eta \leq |s| \to M \models \text{card}(\eta) \leq \alpha$ for some $A \cap \alpha$, $s \restriction \eta$-mouse M. And $\nu_s = \text{ORD}(M_s)$, where M_s is the $<^*$-least core $A \cap \alpha$, s-mouse such that $M_s \models \text{card}(|s|) \leq \alpha$ and ν_s p.r. closed. Note that $t \subseteq s \to M_t$ iterates to an initial segment M_{ts} of M_s via a unique j_{ts}. Then $\nu_s^0 = \bigcup\{\text{ORD}(M_{ts}) \mid t \subseteq s, t \neq s\}$, $s^* = \{\text{ORD}(M_{ts}) \mid t \subseteq s, s(|t|) = 1\}$ and $M_s^0 = L_{\nu_s^0}[\mu_s, A \cap \alpha, s^*]$, where μ_s is the measure of M_s. The fine structure properties of K allow the earlier coding arguments to go through.

If some subset of α does not have a #, then use L-coding. Now notice that there is no conflict between the K-coding and the L-coding, because if $p \in \mathscr{P}^u$, $u \in S_\alpha$ and L-coding is used at α then for CUB-many $\beta < \alpha$, either L-coding is used at β or the ordinals needed to code at β are (beyond some fixed $\beta_0 < \alpha$) larger than those committed by p.

Now suppose κ is Ramsey and $\mathscr{P}$ is the forcing described above. We wish to show that $\mathscr{P} \Vdash \kappa$ is Ramsey. To do so we must make an extra assumption about the predicate A: if α is inaccessible and $\alpha \leq \eta < \alpha^+$ then M collapses to an initial

segment of $\langle L[A], A\rangle$ for CUB-many $M \prec \langle L_\eta[A], A \cap \eta\rangle$. We first "prepare the universe" to create A satisfying this property, while still obeying $H_\gamma = L_\gamma[A]$ for infinite cardinals γ.

$\mathbf{Q}$ (= the forcing to add A) is defined as follows. First fix $A_0 \subseteq \mathrm{ORD}$ such that $H_\alpha = L_\alpha[A_0]$ for infinite cardinals α (and such that the A of the statement of the theorem is coded onto the even part of A_0). Now we force A using conditions $q\colon \mathrm{Dom}(q) \to V$, $\mathrm{Dom}(q)$ a closed initial segment of the cardinals, where $q(\alpha) \in S_\alpha = \{s\colon [\alpha, |s|) \to 2 \mid |s| < \alpha^+,\ L_{|s|}[s, A_0 \cap \alpha] \models \alpha$ is the largest cardinal $\wedge ZF^-$, $L[s, A_0 \cap \alpha] \models |s|$ is not a cardinal$\}$. (Note that we do *not* require $s \in S_\alpha$ to be L-reshaped.) For $s \in S_\alpha$ define $\mathscr{A}^0_s = \langle L_{|s|}[s, A_0 \cap \alpha], s\rangle$ and $\mathscr{A}_s = L_{\mu_s}[s, A_0 \cap \alpha]$, where $\mu_s =$ least p.r. closed $\mu > |s|$ such that $L_\mu[s, A_0 \cap \alpha] \models \mathrm{card}(|s|) \leq \alpha$. Then we also require that $\mathscr{A}_{q(\alpha)} \models \alpha$ inaccessible $\to q \restriction \alpha \in \mathscr{A}_{q(\alpha)}$ and there is CUB $C \subseteq \alpha$, $C \in \mathscr{A}_{q(\alpha)}$, such that $\mathscr{A}^0_{q(\alpha)} =$ direct limit of an elementary chain $\langle \langle \mathscr{A}^0_{q(\beta)} \mid \beta \in C\rangle, \langle \pi_{\beta_\gamma} \mid \beta \leq \gamma$ in $C\rangle\rangle \in \mathscr{A}_{q(\alpha)}$, where π_{β_γ} has critical point β, $\pi_{\beta_\gamma}(\beta) = \gamma$.

Note that if s obeys the first two properties required for membership in S_α then s can be extended to an element of S_α. Moreover given $q \in \mathbf{Q}$ defined on $\mathrm{Card} \cap \alpha^+$, $\mathscr{A}_{q(\alpha)} \models \alpha$ inaccessible, and given $s \in S_\alpha$ extending $q(\alpha)$ we can easily extend q to q' so that $q'(\alpha) = s$. Distributivity is also easily verified. Thus $\mathbf{Q}$ preserves cardinals and adds a predicate B with the strong reflection property: α inaccessible, $\alpha \leq \eta < \alpha^+ \to M \simeq$ an initial segment of $\langle L[B], B\rangle$ for CUB-many $M \prec \langle L_\eta[A], A \cap \eta\rangle$. The desired A is $B \wedge A_0^+$, where $A_0^+ = \{\gamma^+ + \gamma \mid \gamma \in A_0\}$. Note that even for limit α we have $H_{\alpha^+} = L_{\alpha^+}[B]$, so $H_\alpha = L_\alpha[A]$ for all infinite α.

We must also check that $\mathbf{Q}$ preserves Ramseyness. Suppose $q \in \mathbf{Q}$ and κ is Ramsey, and let $I \subseteq \kappa$ consist of good indiscernibles for the structure $\langle L_\kappa[A_0], A_0, q \restriction \kappa\rangle = \mathscr{A}$ and for $\gamma \in I$ let $M_\gamma =$ transitive collapse (Skolem hull of $\gamma \cup \{\gamma_0, \gamma_1, \ldots\}$ in $\mathscr{A}$), where $\gamma_0 < \gamma_1 < \cdots$ are the first ω-many elements of I greater than γ. Then there are natural embeddings $\pi_{\gamma\gamma'}\colon M_\gamma \to M_{\gamma'}$ for $\gamma < \gamma'$ in I. Define μ_γ on $\mathscr{P}(\gamma) \cap M_\gamma$ by $\mu_\gamma(X) = 1$ iff $\gamma \in \pi_{\gamma\gamma'}(X)$ for $\gamma < \gamma'$ in I. Let $\alpha_\gamma = \gamma^+$ in the sense of M_γ. Then $\langle M_\gamma \restriction \alpha_\gamma, \mu_\gamma\rangle$ is amenable for $\gamma \in I$ and direct limit $\langle \langle M_\gamma \restriction \alpha_\gamma, \mu_\gamma\rangle_{\gamma \in I}, \langle \pi_{\gamma\gamma'}\rangle_{\gamma \leq \gamma' \text{ in } I}\rangle$ is of the form $\langle L_\eta[B], B, q, \mu\rangle$, where μ is countably complete. We can extend q to q' so that $|q'| \geq \eta$ and $\{r \mid q' \leq r \leq q, |r| < \eta\}$ reduces each predense $\Delta \subseteq \mathbf{Q} \cap L_\eta[B]$, $\Delta \in L_\eta[B]$ below κ. (Note that $\mathrm{cof}(\eta) = \omega$.)

The point now is that $q' \leq q$ forces that μ extends to a measure on $L_\eta[B, \underline{G}]$. To see this note that $\mathbf{Q} \cap L_\eta[B]$ satisfies μ-distributivity: If $\langle \Delta_\alpha \mid \alpha < \kappa\rangle \in L_\eta[B]$ and Δ_α is predense on $\mathbf{Q} \cap L_\eta[B]$ for $\alpha < \kappa$, then for each $q_0 \in \mathbf{Q} \cap L_\eta[B]$ there is $q_1 \leq q_0$ meeting (μ-measure 1)-many Δ_α's. The latter in fact implies that μ is forced to generate a measure on $L_\eta[B, \underline{G}]$.

Finally suppose $q \Vdash f\colon [\kappa]^{<\omega} \to 2$. We can choose $q' \leq q$ as above so that $q' \Vdash f \in L_\eta[B, \underline{G}]$ and there is a measure μ^* on $L_\eta[B, \underline{G}]$. But then, by countable completeness, $q' \Vdash f$ has a homogeneous set of size κ. So $\mathbf{Q} \Vdash \kappa$ is Ramsey.

Now let us return to $\mathscr{P}$. We want to apply a similar argument (with A_0 replaced by A) to show that $\mathscr{P} \Vdash \kappa$ is Ramsey (when κ is Ramsey in V and hence in $V^{\mathbf{Q}}$). This time we have to choose $q' \leq q$ so that $|q'_\kappa| = \eta$ and q'_κ is R^v-generic, where $v = \bigcup\{j(r)_\eta \mid r \in \mathscr{P}^{q'_\kappa}, |r| < \eta\}$ and $j\colon \langle L_\eta[B], \mu\rangle \to M$ is the ultrapower. Note that in this case $L_\eta[B] = L_\eta[A]$ and $B \cap \eta = A \cap \eta$, thanks to our preparatory forcing to add A. Also η is singular in $K^{A \cap k, A \cap \eta}$ and hence in $K^{A \cap \kappa, q'_\kappa}$ as required for $q'_\kappa \in S_\kappa$. We must also know that q' could have been chosen to avoid the restraint

imposed by $\bar{q}_\kappa$. But the A-restraint in $\bar{q}_\kappa$ is appropriate for the forcing R^v. To avoid the coding restraint in $\bar{q}_\kappa$, observe that this restrains strings t such that $\alpha(t)$ is A-stable in κ^+:

$$\langle L_{\alpha(t)}[A \cap \alpha(t)], A \cap \alpha(t)\rangle \prec \langle L_{\kappa^+}[A], A \cap \kappa^+\rangle.$$

We may assume that the extension $q_\kappa \subseteq q'_\kappa$ takes place between adjacent A-stables in κ^+. So there is no conflict.

Once again we get that $q \Vdash f\colon [\kappa]^{<\omega} \to 2$ implies $q' \Vdash f$ has a size κ homogeneous set for some $q' \leq q$, and thus $\mathscr{P} \Vdash \kappa$ is Ramsey. $\dashv$

REMARK. Implicit in the previous proof is the following fact, due to a number of people.

PROPOSITION 4.3. *Assume GCH. Then κ is Ramsey iff for every $A \subseteq \kappa$ there exists μ, M such that $\langle M, \mu\rangle$ is amenable, $A \in M$, $\langle M, \mu\rangle \models \mu$ is a measure on κ and μ is countably complete.*

Also note that by the method of Jensen [68], we can drop the GCH assumption in Theorem 4.2 if we also drop the requirement of cardinal preservation.

The next result is analogous to a result of Beller and David (see §5.2 of Beller, Jensen and Welch [82], or David [82]). M is a ZF, R-mouse (for $R \subseteq \omega$) if $M = \langle L_\alpha[\mu, R], \mu\rangle$ is an R-mouse satisfying ZF. If $R = \varnothing$ we say that M is a ZF-mouse.

THEOREM 4.4. *Suppose M is a countable ZF-mouse. Then there exists a real R such that $M[R]$ is the $<^*$-least ZF, R-mouse. Moreover the measure of $M[R]$ extends the measure of M.*

PROOF. Suppose $M = \langle L_\alpha[\mu], \mu\rangle$, μ a measure on κ. First produce an L-reshaped $A \subseteq (\kappa^+)^M$ such that no $\beta < \alpha$ satisfies $L_\beta[\mu, A \cap \xi] \models \mathrm{ZF} + \xi = \kappa^+$ for any ξ. This is possible using the proof of Theorem 5.2 in Beller, Jensen and Welch [82] or David [82]. Then code A by a real, preserving the measurability of κ. If $N[R]$ is a ZF, R-mouse then compare it to $M[R]$; if the iterate of $N[R]$ is a proper initial segment of the iterate $M^*[R]$ of $M[R]$, then A^* = image of A in $M^*[R]$ fails to obey in $M^*[R]$ the defining property of A in $M[R]$, contradiction. $\dashv$

THEOREM 4.5. *The existence of a precipitous ideal on ω_1 is consistent with the existence of a Δ^1_3 well-ordering of the reals.*

PROOF. Start with $L[\mu]$, μ a measure on κ, and collapse κ to ω_1 in a special way: Use κ^{++}-Souslin trees from $L[\mu]$ as in David [83] to guarantee that each successor cardinal $< \kappa$ will have a canonical real code. Thus we $L[\mu]$-code branches through the trees into subsets of cardinals $\alpha^{++} < \kappa$ and then define a_α to collapse α^+ and to be uniquely determined by the subsets of α^{++} that it almost disjointly codes. Now notice that the relation "R codes an ordinal $\alpha \geq (\omega_1)^{K^S}$" is a π^1_2 relation on reals: it holds iff whenever M is an S-mouse, $M \models |R|$ is uncountable iff whenever M is a transitive model of $\mathrm{ZF}^- + (V = L[\mu, S]) + (\mu$ is a measure), $|R| \in M \to M \models |R|$ is uncountable. The latter equivalence follows as the hypothesis $|R| \in M$ guarantees that M is an iterable S-premouse when $|R| \geq (\omega_1)^{K^S}$.

Thus we have that there is in $L[\mu, \langle a_\alpha \mid \alpha < \kappa\rangle]$ a π^1_2 relation $P(R, S) \leftrightarrow S = \langle a_\beta \mid \beta \leq \alpha\rangle, \alpha \geq (\omega_1)^{K^R}$. We can now well-order the reals by $R_0 \leq R_1 \leftrightarrow \exists S(P(R_1, S)$ and $R_0 \leq R_1$ in $K^S)$. This is Δ^1_3.

To get a precipitous ideal, follow Jech, Magidor, Mitchell and Prikry [80] and take $I = \{X \subseteq \kappa \mid \exists Y(\mu(Y) = 1$ and $X \cap Y = \varnothing)\}$. The proof that I is precipitous goes through, using the above forcing in place of the gentle Lévy collapse of κ. $\dashv$

Some open questions. 1) Is there a coding theorem for hypermeasures?

2) Suppose $0^{\dagger} \notin L[\mu, R]$ and μ is a measure, $R \subseteq \omega$. Then is R generic over $L[\mu]$ (via possibly a class forcing)?

3) Say that M is recursively inaccessible if M is admissible and $x \in M \to x \in y \in M$ where y is admissible. Is there a real R such that $M[R]$ is an admissible R-mouse iff M is a recursively inaccessible mouse?

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