The Hyperuniverse Programme

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Truth in Modern Set Theory

Early 20th Century situation: "Resolution" of the paradoxes, analysis of the "iterative concept of set" leading to the formulation of

ZFC = Zermelo-Fraenkel Set Theory with AC (Axiom of Choice)

"The standard axioms"

Infinity (ω exists)
Powerset ($P(x) = \{y \mid y \subseteq x\}$ exists)
Replacement ($F[x]$ exists if F is a definable operation)

The Hyperuniverse Programme: A Standard Picture

"A standard picture of V (the universe of all sets)"

Ordinals
$0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega + \omega, \ldots, \omega_1, \ldots, \omega_2, \ldots$

The von Neumann Hierarchy
$V_0 \subseteq V_1 \subseteq \cdots V_\omega \subseteq V_{\omega+1} \subseteq \cdots V_{\omega+\omega} \subseteq \cdots$ with limit V

where $V_{\alpha+1} = P(V_\alpha)$

V is a two-parameter structure, determined by the *ordinals* and the *powerset operation*.

The Hyperuniverse Programme: Extrinsic and Intrinsic

ZFC is the "standard theory" for two reasons:

Extrinsic Reason: In its role as a foundation for mathematics, ZFC is very effective, i.e., nearly all theorems of mathematics can be easily translated into set theory and derived from ZFC

Intrinsic Reason: The axioms of ZFC can be derived from the "iterative concept" of set, by which sets are generated by unlimited iterations of the powerset operation through the ordinal numbers (as in the "standard picture of V")

[Apology: Not everyone agrees with the latter claim. However my emphasis in this talk will not be on justifications of the ZFC axioms, but rather on justifications for new axiom candidates beyond ZFC.]

The Hyperuniverse Programme: Extrinsic and Intrinsic

In other words, there are both *extrinsic* and *intrinsic* reasons for asserting:

The axioms of ZFC are true

But there are many important questions in set theory which are not resolvable just using the axioms of ZFC. For example:

CH (Continuum Hypothesis): All uncountable sets of real numbers have the same size.

Key Motivating Question: Are there both *extrinsic* and *intrinsic* reasons for asserting the truth of axiom-candidates not derivable from ZFC? If so, are there such axiom-candidates which resolve important problems like CH, which are not resolvable in ZFC alone?

The Hyperuniverse Programme: Reflection

In fact the "iterative concept of set" can give us a bit more than ZFC.

The Reflection Principle: If a "property" holds of V then it holds of some V_α.

This is derivable from the iterative concept as it simply means that if one can iterate the powerset operation long enough to reach a V_α satisfying some property then one can iterate the powerset operation further.
[Same Apology as before.]

Reflection implies for example that not only V, but also some V_α is a model of ZFC. This is not derivable from the ZFC axioms.

However even though Reflection takes us past ZFC, it is rather weak, without major consequences. It is consistent with $V = L$.

The Hyperuniverse Programme: Extrinsic and Intrinsic

A preview of what is to come:

1. There is substantial *extrinsic evidence* coming from *set theory* for a number of axiom-candidates with important consequences.

2. There is a current lack of *extrinsic evidence* coming from *other areas of mathematics or logic* for axiom-candidates with important consequences.

3. The *Hyperuniverse Programme* is a new source of *intrinsic evidence* for axiom-candidates with important consequences.

The Hyperuniverse Programme: Extrinsic evidence from Set Theory

Probably the best-known extrinsic evidence for new axioms results from the power of large cardinal axioms (strong axioms of infinity) to resolve questions in descriptive set theory.

Descriptive set theory is concerned with the *projective* sets of real numbers, obtained by closing the open sets under continuous images and complements. The projective sets are organised into a hierarchy

$$\Sigma^1_1 \subseteq \Sigma^1_2 \subseteq \cdots$$

with each projective set appearing at some level.

Now here is the evidence:

The Hyperuniverse Programme: Extrinsic evidence from Set Theory

1. ZFC proves that Σ^1_1 sets have some nice properties: PSP (perfect set property), LM (Lebesgue meaurability), PB (Property of Baire).

2. ZFC + large cardinal axioms (but not ZFC alone) proves that Σ^1_n sets have these nice properties for all n.

Thus we have extrinsic evidence from set theory for the truth of large cardinal axioms.

The Hyperuniverse Programme: Extrinsic evidence from Set Theory, Criticism

There are however some problems with this extrinsic evidence for the truth of large cardinal axioms:

Problem 1. The argument is based on an extrapolation from Σ^1_1 to Σ^1_n for all n. But there are simple examples of analagous extrapolations that lead to contradiction: For example, even though Σ^1_2 absoluteness for arbitrary models $M \subseteq N$ of ZFC with the same ordinals is ZFC-provable, Σ^1_3 absoluteness is ZFC-provably false!

Problem 2. Large cardinal axioms are often formulated in terms of the existence of embeddings $j : V \to M$ of the universe V of sets into a transitive inner model M of ZFC which approximates V. By taking this to its natural conclusion, requiring $M = V$, one arrives at a contradiction. Thus large cardinal axioms are not "stable".

The Hyperuniverse Programme: Extrinsic evidence from Set Theory

Another extrinsic source for new axioms comes from *Set-Generic Absoluteness Principles*:

A set is *transitive* if it contains the elements of its elements.

For any infinite cardinal number κ, $H(\kappa)$ denotes the union of all transitive sets of size less than κ.

In particular, $H(\omega) =$ the union of all finite transitive sets, $H(\omega_1) =$ the union of all countable transitive sets and $H(\omega_2) =$ the union of all transitive sets of size ω_1, the least uncountable cardinal number.

We write $M \sqsubseteq N$ if $M \subseteq N$ are transitive models of ZFC with the same ordinals.

The Hyperuniverse Programme: Extrinsic evidence from Set Theory

Trivial absoluteness: If $M \sqsubseteq N$ are models of ZFC then the theory of $H(\omega)$ is the same in M and N.

Can we replace $H(\omega)$ by $H(\omega_1)$ or even $H(\omega_2)$?

$M \sqsubseteq^{set-generic} N$ iff $M \sqsubseteq N$ *and* N *is a set-generic extension of* M
$M \sqsubseteq^{stat-pres-set-generic} N$ iff $M \sqsubseteq N$ *and* N *is a stationary-preserving set-generic extension of* M

Woodin set-generic absoluteness: If $M \sqsubseteq^{set-generic} N$ are models of ZFC + large cardinals then the theory of $H(\omega_1)$ is the same in M and N.

Viale stationary-preserving set-generic absoluteness: If $M \sqsubseteq^{stat-pres-set-generic} N$ are models of ZFC + large cardinals + MM^{+++} then the theory of $H(\omega_2)$ is the same in M and N.

The Hyperuniverse Programme: Extrinsic evidence from Set Theory, Criticism

Thus via Absoluteness Principles we have extrinsice evidence from set theory for large cardinal axioms together with the “forcing axiom” MM^{+++}.

Again there is a problem with this type of argument:

Problem 3. The Woodin and Viale Absoluteness Principles are based on set-generic extensions $M \sqsubseteq^{set-generic} N$. If one allows more general extensions then these principles become inconsistent. Consider the view of Paul Cohen, the inventor of set-genericity:

“Cohen said that he was surprised to see that generic extensions, or forcing extensions, were being used as fundamental notions in their own right, rather than just technical artifacts of his (Cohen’s) method of proof.”

The Hyperuniverse Programme: Extrinsic evidence from Mathematics?

For these reasons it is worthwhile to look for other sources of evidence, either extrinsic evidence from other areas of mathematics or evidence of an intrinsic nature.

A systematic study of what axioms of set theory are most effective for other areas of logic and mathematics is yet to be undertaken. There are indications from model theory that weak forms of the GCH are needed, but that is a very preliminary conclusion. It is not yet known what the needs are of topology, functional analysis and homological algebra, where undecidable problems often arise.

So instead we look for *intrinsic evidence*.

Until now there has been very little progress on this question, for a number of reasons:

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Feferman's vagueness argument:
"I have been led to the view that the statement CH is inherently vague and that it is meaningless to speak of its truth value; the fact that no remotely plausible axioms of higher set theory serve to settle CH only bolsters my conviction."

Shelah's pluralist view:
"My feeling is that ZFC exhausts our intuition except for things like consistency statements, so a proof means a proof in ZFC."
"I do not feel a universe of ZFC is like the sun, it is rather like a human being of some fixed nationality."

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Balaguer's full-blooded Platonism (FBP):
"According to FBP, both ZFC and ZF+ not-C [negation of AC] truly describe parts of the mathematical realm; but there is nothing wrong with this, because they describe different parts of that realm. This might be expressed by saying that ZFC describes the universe of sets1, while ZF+not-C describes sets2, where sets1 and sets2 are different kinds of things."

"What FBP says is that there are so many different kinds of sets that every consistent theory is true of an actual universe of sets."

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Hamkins goes even further:
"... the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem ... the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question."

Maddy's naturalism:
"What, then, does naturalism suggest for the case of the CH? First, that we needn't concern ourselves with whether or not the CH has a determinate truth value ... Instead, we need to assess the prospects of finding a new axiom that is well-suited to the goals of set theory and also settles CH."

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

We come now to *The Hyperuniverse Programme*

We have seen that there are axioms derivable from the "iterative concept of set" which take us beyond ZFC, such as reflection principles. However it has proved very difficult to obtain much more than that using intrinsic evidence *based on the concept of set*.

The Hyperuniverse Programme instead focuses on intrinsic evidence based on the concept of *universe of sets*, and derives first-order consequences from this.

The Programme can be outlined as follows:

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 1. Create a context in which different pictures of V (universes) can be compared, the *Hyperuniverse*.

Step 2. The comparison of universes evokes *intrinsic principles*, such as maximality, for the choice of "preferred universes".

Step 3. These intrinsic principles are then formulated mathematically as specific *mathematical criteria* for the selection of preferred universes.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 4. Each criterion gives rise to the collection of *preferred universes* which satisfy it, and first-order statements which hold in all such preferred universes are proposed as *axiom-candidates*.

Step 5. The axiom-candidates following from a given criterion are then *tested* according to their compatibility with set-theoretic practice and, ideally, for the existence of extrinsic evidence for them.

Step 6. (The ultimate goal) If the axiom-candidates following from a given criterion are compatible with set-theoretic practice and, ideally, if there is extrinsic evidence for them, then they are proposed as *new and true axioms of set theory*.

Let us take now a closer look at these steps and accompany this with a report on the developments within the Hyperuniverse Programme to date.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 1. Create a context in which different pictures of V (universes) can be compared, the *Hyperuniverse*.

The Hyperuniverse consists simply of the countable transitive models of ZFC. There are very strong reasons in favour of this choice:

a. V, the universe of all sets, is itself a transitive model of ZFC, so by taking our "pictures of V" to also be transitive models of ZFC we are remaining faithful to this key aspect of V.

b. By restricting ourselves to *countable* models we have lost none of the possibilities for *first-order* properties of V, as any first-order property of V must hold in a countable transitive model.

c. The collection of countable transitive models of ZFC is closed under all known methods for producing new (well-founded) models of ZFC from old ones, including the methods of forcing and infinitary logic.

A key point: We do not take a Platonist position, i.e., we do not regard V as a fixed and well-determined class of objects. Instead, our concept of V is *epistemic* and *dynamic*, whereby via the Hyperuniverse Programme we clarify our understanding of V by exploring preferred pictures of it. Nor do we have a Platonist view of the Hyperuniverse, and indeed there is a dynamic interaction between our understanding of V and our understanding of the Hyperuniverse.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 2. The comparison of universes evokes *intrinsic principles*, such as maximality, for the choice of "preferred universes".

We regard maximality as an *intrinsic feature* of the universe of sets. But our treatment of maximality goes beyond what is derivable from the iterative concept, namely, what sets must exist. The Hyperuniverse Programme allows us to *compare* different pictures of V and thereby isolate pictures of V which are "maximal" with regard to this comparison.

Another principle which we consider is *omniscience*, which asserts that although (by Tarski) one cannot define what it means to be true in V, one can nevertheless define what it means to be true in some universe that contains V (a Gödel-like completeness theorem).

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 3. These intrinsic principles are then formulated mathematically as specific *mathematical criteria* for the selection of preferred universes.

The principle of Maximality has been formulated mathematically in a number of different ways:

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

The Inner Model Hypothesis (IMH) or Powerset Maximality: Let $\Phi(V)$ denote the set of sentences which hold in some inner model $M \sqsubseteq V$. Then if $V \sqsubseteq W$ we have $\Phi(V) = \Phi(W)$.

#-Generation or Ordinal Maximality: The universe V is #-generated (not defined here). We regard this as the strongest possible form of Reflection.

Omniscience: The set of first-order sentences with parameters from V which hold in some $W \sqsupseteq V$ is definable in V.

The above have all been shown to be consistent, i.e. to hold in some element of the Hyperuniverse (assuming the consistency of large cardinals).

The Strong IMH: The IMH for sentences with *absolute parameters*.

This is conjectured to be, but not known to be, consistent.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Sometimes criteria are combined to arrive at *synthesised criteria*.

Some examples:

The IMH for #-generated universes

#-generation together with Omniscience

These have been shown to be consistent.

The IMH for omniscient universes or for universes which are both #-generated and omniscient

This has only been shown to be consistent if one adds to it the existence of a proper class of measurable cardinals.

The Strong IMH for for universes which are both #-generated and omniscient

This is currently the "strongest" criterion, not yet known to be consistent.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 4. Each criterion gives rise to the collection of *preferred universes* which satisfy it, and first-order statements which hold in all such universes are proposed as *axiom-candidates*.

Some axiom-candidates that have arisen in this way are:

1. (Consequences of the IMH) There are no inaccessibles and some Σ^1_3 set of reals is not Lebesgue measurable. But there are inner models with measurables.

2. (Consequence of #-generation) There exist inaccessible (even weakly compact) cardinals.

(Note that 1 and 2 contradict each other.)

3. (Conjectured consequence of Omniscience) There are inner models with Ramsey cardinals.

4. (Consequence of the Strong IMH and its synthesised versions) CH is false, indeed the cardinality of the continuum is very large.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 5. The axiom-candidates following from a given criterion are then *tested* according to their compatibility with set-theoretic practice and, ideally, for the existence of extrinsic evidence for them.

Regarding the current axiom-candidates, listed above:

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

1. There are no inaccessibles and some Σ^1_3 set of reals is not Lebesgue measurable. But there are inner models with measurables.
2. There exist inaccessible (even weakly compact) cardinals.
3. There are inner models with Ramsey cardinals.
4. CH is false, indeed the cardinality of the continuum is very large.

All but 1 are compatible with set-theoretic practice; Arrigoni and I argue that even 1 might be compatible if one re-examines the roles of large cardinals and axioms of determinacy in set theory.

For 3 there is substantial extrinsic evidence.

The most interesting is 4, for which there is perhaps some extrinsic evidence.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Step 6. (The ultimate goal) If the axiom-candidates following from a given criterion are compatible with set-theoretic practice and, ideally, if there is extrinsic evidence for them, then they are proposed as *new and true axioms of set theory*.

The current situation is the following:

We have intrinsic evidence for the existence of weakly compact cardinals and this is compatible with set-theoretic practice.

Subject to a conjecture about omniscience, we have both extrinsic and intrinsic evidence for the existence of inner models with Ramsey cardinals.

Subject to the consistency of (synthesised forms of) the Strong IMH, we have intrinsic evidence, compatible with set-theoretic practice, for the failure of CH.

The Hyperuniverse Programme: Intrinsic Evidence for Set-Theoretic Truth

Thus the Hyperuniverse Programme, which is still very young, is pointing towards new and true axioms of set theory asserting the existence of "small" large cardinals (weakly compact), the existence of inner models with much bigger large cardinals (hypermeasurable and beyond), as well as a strong failure of CH.

But a huge amount of work needs to be done, both on the mathematical side, verifying the consistency of various criteria, as well as on the philosophical side, justifying the claim that principles like maximality and omniscience are intrinsic to our concept of set-theoretic universe.

I am excited to learn how things turn out.

Postscript: Relation to Peter Koellner's Talk

As Peter and I discussed similar topics perhaps it would be useful to briefly delineate some of the differences between our points of view:

1. I agree with Peter's distinction between *intrinsic* (a priori) and *extrinsic* (a posteriori) evidence. But whereas Peter suggests that intrinsic evidence is limited to reflection, I propose a new form of intrinsic evidence based on the concept of *set-theoretic universe* which has consequences beyond reflection.

2. I do not support the claim that the consistency of large cardinal axioms entails their existence. The former is justified by the existence of inner models for large cardinals whereas the latter is not. But I agree that the claim is correct for statements about V_ω, such as the totality of exponentiation, because V_ω has no proper inner models. And I agree that there is strong evidence for very large cardinals to exist *in inner models*.

Postscript: Relation to Peter Koellner's Talk

3. In my view the extrinsic evidence for AD in $L(R)$ is not convincing. In particular I don't subscribe to the intrinsic plausibility of regularity properties for the higher projective levels, as this is based on an extrapolation from the first projective level; such extrapolations are known to fail for absoluteness principles and for regularity properties in generalised Baire Space. Moreover, I do not agree that AD in $L(R)$ is a consequence of all strong theories; a counterexample is the theory asserting the existence of inner models with supercompact cardinals. But in light of the strong evidence that large cardinals exist in inner models, we can conclude that AD holds in some inner model (that may fail to contain all reals).

4. As Cohen himself suggested, I do not think that forcing (whether it be set-forcing, class-forcing or hyperclass-forcing) can play a legitimate role in discussions of evidence for new axioms of set theory. In particular Ω-logic cannot play such a role.

Postscript: Relation to Peter Koellner's Talk

In conclusion, I feel that a modification of the kind of *extrinsic* evidence that Peter provided bolstered by further *intrinsic* evidence such as provided by the Hyperuniverse Programme is needed to establish the truth of new axiom-candidates for set theory. This is a very intriguing prospect, for which a great deal of work remains to be done.