Cardinal-Preserving Extensions

Sy D. Friedman *
University of Vienna and MIT

May 19, 2003

Abstract A classic result of Baumgartner-Harrington-Kleinberg [1] implies that assuming CH a stationary subset of ω_1 has a CUB subset in a cardinal-perserving generic extension of V, via a forcing of cardinality ω_1. Therefore, assuming that ω_2^L is countable: $\{X \in L \mid X \subseteq \omega_1^L$ and X has a CUB subset in a cardinal-preserving extension of $L\}$ is constructible, as it equals the set of constructible subsets of ω_1^L which in L are stationary. Is there a similar such result for subsets of ω_2^L? Building on work of M. Stanley [9], we show that there is not. We shall also consider a number of related problems, examining the extent to which they are "solvable" in the above sense, as well as defining a notion of reduction between them.

We assume throughout that $0^\#$ exists.

Definition A subset X of L is Σ_1^{CP} iff X can be written in the form

$$a \in X \text{ iff } \varphi(a) \text{ holds in a cardinal-preserving extension of } L$$

for some Σ_1 formula φ. We intend our cardinal-preserving extensions of L to satisfy AC and to be contained in a set-generic extension of V. (In all natural cases, the words "a set-generic extension of" can be omitted; see the Remark following the proof of Theorem 1 below.)

Theorem 1 *If X is Σ_1^{CP} then X is constructible from $0^\#$.*

*Math. Subj. Class. 03E35, 03E45, and 03E55. Keywords and phrases: Descriptive set theory, large cardinals, innermodels. The author wishes to thank the Mittag-Leffler Institute, for its generous support during September, 2000, when this paper was written. This research was generously supported by Project P13983 of the Austrian Fonds zur Förderung der wissenschaftlichen Forschung (FWF).

Theorem 2 $0^\#$ *is* Σ_1^{CP}*. And there are* Σ_1^{CP} *sets of constructibility degree strictly between* 0 *and* $0^\#$*.*

Theorem 3 *The following* Σ_1^{CP} *sets are equiconstructible with* $0^\#$*:*
(a) $\{T \mid T \in L$ *and* T *is a tree on* κ *of height* κ *with a cofinal branch in a cardinal-preserving extension of* $L\}$*, for* κ *an infinite successor cardinal of* L*.*
(b) $\{X \subseteq \kappa \mid X \in L$ *and* X *contains a CUB subset in a cardinal-preserving extension of* $L\}$*, for* κ *regular in* L*,* $\kappa > \omega_1^L$*.*
(c) $\{X \subseteq \kappa \mid X \in L$ *and* X *is the set of ordinals* $< \kappa$ *which are admissible relative to some real in a cardinal-preserving extension of* $L\}$*, for* κ *an uncountable cardinal of* L*.*
(d) $\{X \subseteq \kappa \mid X \in L$ *and* X *is the intersection with* κ *of a class which is* Δ_1*-definable over* $L[R]$ *without parameters, for some real* R *in a cardinal-preserving extension of* $L\}$*, where* κ *is an* L*-cardinal greater than* ω_2^L*.*

Theorem 3 will be proved by "reducing" $0^\#$ to the sets mentioned. In fact we shall need the following more general notion of "reduction".

Definition Suppose that (X_0, X_1) and (Y_0, Y_1) are pairs of disjoint subsets of L. Then we write
$(X_0, X_1) \longrightarrow_L (Y_0, Y_1)$
iff there is a function F in L such that
$a \in X_0 \to F(a) \in Y_0$
$a \in X_1 \to F(a) \in Y_1$.

We write X_0 instead of (X_0, X_1) in case X_0 is the complement (within some constructible set) of X_1, and similarly for the Y's. It is clear that if (X_0, X_1) is nonconstructible, $X_0 \cup X_1$ is constructible and $(X_0, X_1) \longrightarrow_L (Y_0, Y_1)$, then (Y_0, Y_1) is also nonconstructible. In the proof of Theorem 3 we shall obtain reductions in this sense of $0^\#$ to the sets mentioned.

Theorem 3 suggests that the Baumgartner-Harrington-Kleinberg result should be viewed as a rare example of a nontrivial "solvable" Σ_1^{CP} problem. However it is not the only such example:

Theorem 4 *If* κ *is* ω_2^L *in the set described in Theorem 3 (d), then the resulting set is constructible.*

Proof of Theorem 1. Suppose that X is Σ_1^{CP} and therefore can be written in the form

$$a \in X \text{ iff } \varphi(a) \text{ holds in a cardinal-preserving extension of } L$$

where φ is Σ_1.

Suppose that a belongs to X and choose a set of ordinals S in a set-generic extension $V[G]$ of V such that $\varphi(a)$ is true in $L[S]$ where $L[S]$ is a cardinal-preserving extension of L. Choose α so that $X \subseteq L_\alpha$, $S \subseteq \alpha$ and the parameters in φ belong to L_α. Let c be a real generic over $V[G]$ for a constructible set-forcing such that α and the parameters in φ are coded by c. The set S is countable in $V[G][c]$. Thus

$Y_a = \{b \in V[G][c] \mid b$ is a real and b codes both a set of ordinals S such that $L[S]$ is a cardinal-preserving extension of L satisfying $\varphi(a)$, together with a witness for $\varphi(a)\}$

is nonempty. Moreover Y_a is Π^1_2 in c and has an element b constructible from the pair c, S. As c is set-generic over $V[G]$ and therefore over $L[S]$, the pair b, c does not construct $0^\#$. It follows from the Main Lemma of Harrington-Kechris [6] that Y_a has an element b in $L[c^\#] = L[0^\#, c]$, and the latter is a set-generic extension of $L[0^\#]$. As this applies to any element a of X, we have:

$a \in X$ implies
$\varphi(a)$ is true in a cardinal-preserving extension of L included in a set-generic extension of $L[0^\#]$.

Conversely, if $\varphi(a)$ is true in a cardinal-preserving extension of L included in a P-generic extension $L[0^\#]$ for some forcing $P \in L[0^\#]$, then this fact is forced by some condition $p \in P$ and therefore $\varphi(a)$ is true in a cardinal-preserving extension of L included in a P-generic extension of V. Thus the above implication is in fact an equivalence, giving $X \in L[0^\#]$. □

Remark. Suppose that X is Σ_1^{CP} via the formula φ, X is a subset of L_α for some countable ordinal α and the parameters in φ are countable. Then the previous proof shows that if a belongs to X, then $\varphi(a)$ has a solution in a cardinal-preserving extension of L *contained in* V. This is because in V

there exist generics for any countable constructible forcing. It follows that in this special case, we can equivalently define Σ_1^{CP} using cardinal-preserving extensions of L which exist in V, and not only in a set-generic extension of V.

Proof of Theorem 2. Notice that any set of the form

$n \in X$ iff $\exists R(\varphi(n, R) \wedge \mathrm{Card}^{L[R]} = \mathrm{Card}^L)$,

where φ is Π_2^1 without parameters, is Σ_1^{CP}, as for reals R that preserve L-cardinals, $\varphi(n, R)$ holds exactly if it holds in $L_{\omega_1^L}[R]$. So to show that $0^\#$ is Σ_1^{CP} it suffices to show that it can be written in the above form.

In [3] a notion of "guess" at a finite sequence of Silver indiscernibles was defined, together with a notion of "killing a guess". It was shown that actual finite sequences of Silver indiscernibles cannot be killed, but there are reals that preserve cardinals over L which kill all guesses which "explicitly contradict" $0^\#$. Using these methods one can show:

If n does belong to $0^\#$ then there is a real R which preserves cardinals over L such that R kills every guess which says that n does not belong to $0^\#$.

As finite sequences of actual Silver indiscernibles cannot be killed, it follows that $n \in 0^\#$ iff there is a real R which preserves cardinals over L such that R kills every guess which says that n does not belong to $0^\#$, and therefore $0^\#$ is Σ_1^{CP}.

In [3] it was also shown that there is a real R such that:
(1) R is nonconstructible and preserves L-cardinals.
(2) $n \in R$ iff R kills every guess which says that n does not belong to R iff there exists a real S such that S preserves L-cardinals and S kills every guess which says that n does not belong to S.

It follows that R is Σ_1^{CP}, and by (1), R has L-degree strictly between 0 and $0^\#$. $\square$

Proof of Theorem 3. (a) For κ regular in L we let $\mathcal{T}(\kappa)$ denote $\{T \mid T \in L$ and T is a tree on κ of height κ with a cofinal, cardinal-preserving branch$\}$, where in general a set a is *cardinal-preserving* if L and $L[a]$ have the same cardinals. It is sufficient to show that $0^\# \longrightarrow_L \mathcal{T}(\kappa)$, when κ is the L-successor to an infinite L-cardinal λ. For this purpose we again use the "guess-killing" method of [3]. Let T_n consist of all $s \in L, s : |s| \to 2, \lambda \leq |s| < \kappa$ such that:

$(*)$ For all $\lambda < \eta \leq |s|$, if $L_\beta[s \restriction \eta] \models ZF^- + \eta = \lambda^+$ then $L_\beta[s \restriction \eta] \models$ Every guess which says that n does not belong to $0^\#$ has been killed.

Using the methods of [3] it can be shown that T_n has a cofinal cardinal-preserving branch when n belongs to $0^\#$. But note that by reflection, if T_n has a cardinal-preserving branch b then in $L[b]$ every guess which says that n does not belong to $0^\#$ has been killed, and therefore n really does belong to $0^\#$. It follows that $n \in 0^\#$ iff T_n belongs to $\mathcal{T}(\kappa)$, as desired.

(b) We first improve the result of part (a). Let κ be the L-successor to a regular L-cardinal λ. Define:

$\mathcal{T}^*(\kappa) = \{T \in \mathcal{T}(\kappa) \mid T$ is Δ_1-definable over L_κ from the parameter λ and T has a cardinal-preserving, stationary, $\mathcal{P}(\lambda)$-preserving cofinal branch$\}$
$\mathcal{T}^{**}(\kappa) = \{T \in \mathcal{T}(\kappa) \mid T$ is Δ_1-definable over L_κ from the parameter λ and T has a cardinal-preserving cofinal branch$\}$,

where $b \subseteq \kappa$ is *stationary* if in $L[b]$ its intersection with $(\text{cof } \lambda)^L$ is stationary, and b is $\mathcal{P}(\lambda)$*-preserving* if L and $L[b]$ have the same subsets of λ. We first show: $0^\# \longrightarrow_L (\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L))$. Let T_n, $n \in \omega$ be the trees defined in the proof of (a), when $\kappa = \omega_2^L$. Then n belongs to $0^\#$ iff T_n has a cardinal-preserving, $\mathcal{P}(\omega_1^L)$-preserving, cofinal branch. Moreover, if P_n is the L-definable forcing for producing such a branch, then for n in $0^\#$ this branch can be required to satisfy the stronger version of $(*)$ in which we require $s \restriction \eta$ to code over L_β a generic for $(P_n)^{L_\beta}$.

We aim to define a new sequence of trees $T_n^*, n \in \omega$. To do so we first define $f_n(\alpha)$ by induction on $\alpha < \omega_2^L$:
(1) If $f_n(\bar{\alpha})$ has length $\geq \alpha$ for some $\bar{\alpha} < \alpha$ or L-cof $(\alpha) = \omega$ then $f_n(\alpha)$ is the least element of T_n of length $\geq \alpha$ not in $f_n[\alpha]$.
(2) If (1) fails then let β be least such that $L_\beta \models \mathrm{ZF}^- + \alpha = \omega_2$ and for some least condition $p \in (P_n)^{L_\beta}$ and least name σ in $L_\beta : p \Vdash \sigma$ is a CUB subset

of α disjoint from $f_n[b^G]$, b^G denoting the generic branch through $T_n \upharpoonright \alpha$. If β does not exist then define $f_n(\alpha)$ as in (1). Otherwise choose $f_n(\alpha)$ to be the least element of T_n of length α coding a $(P_n)^{L_\beta}$-generic extending p.

Now define T_n^* by $\alpha <_{T_n^*} \beta$ iff $f_n(\alpha) <_{T_n} f_n(\beta)$. Suppose that b is a $\mathcal{P}(\omega_1^L)$-preserving branch through T_n. We claim that $f_n[b] \cap (\mathrm{cof}\ \omega_1)^L$ is a stationary branch through T_n^*. If not then (as b generically codes a P_n-generic) there is $\beta > \omega_2^L$ satisfying $L_\beta \vDash \mathrm{ZF}^-$, $p \in (P_n)^{L_\beta}$ and $\sigma \in L_\beta$ such that $p \Vdash \sigma$ is a CUB subset of ω_2^L disjoint from $f_n[b^G]$, where b^G denotes the generic branch through T_n. Choose β, p, σ to be least. By reflection there is $\alpha \in (\mathrm{cof}\ \omega_1)^L \cap \omega_2^L$ so that $f_n(\alpha)$ is defined as in the last part of (2) above. Choose c to be a branch through T_n generically coding a $(P_n)^{L_\beta}$-generic extending p, where c extends $f_n(\alpha)$. (This is possible, using the countability of β in $L[0^\#]$, as we can choose α so that $\alpha = \omega_2^L \cap$ Skolem hull (α) in L_β, thereby enabling the extension from p_{ω_1} to $f_n(\alpha)$ to obey the restraint imposed by $p^*_{\omega_1}$). But now we have a contradiction, as α is a limit of σ^c, a CUB subset of ω_2^L, and $f_n(\alpha)$ lies on c.

If n belongs to $0^\#$ then T_n has a generic branch and hence a cardinal-preserving, $\mathcal{P}(\omega_1^L)$-preserving branch; by the above, T_n^* has a stationary such branch. If n does not belong to $0^\#$ then T_n has no cardinal-preserving branch and hence T_n^* has no cardinal-preserving branch, as such a branch b would yield the cardinal-preserving branch $f_n[b]$ through T_n. Thus it follows that $0^\# \longrightarrow_L (\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L))$.

The next lemma is from [9], reformulated in terms of our notion of reduction.

Lemma 5 $(\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L)) \longrightarrow_L \mathcal{C}(\omega_2^L)$, *where for L-regular* κ, $\mathcal{C}(\kappa) = \{X \subseteq \kappa \mid X \in L$ *and* X *contains a cardinal-preserving CUB subset*$\}$.

Proof of Lemma 5. Let T be a tree on ω_2^L of height ω_2^L which is Δ_1-definable over $L_{\omega_2^L}$ from the parameter ω_1^L. Then there is a gap 1 morass at ω_1^L in L such that:

(1) To each $\sigma \in \cup\{S_\alpha \mid \alpha \leq \omega_1^L\}$ is associated a tree T_σ on σ.
(2) For $\sigma \in S_{\omega_1^L}$, $T_\sigma = T \upharpoonright \sigma$.
(3) For $\sigma <_1 \tau$, $T_\sigma = \pi_{\sigma\tau}^{-1}[T_\tau]$ and for $\sigma <_0 \tau$, $T_\sigma = T_\tau \upharpoonright \sigma$.

(Our notation is as follows: S_α is the α-th level of the morass; $\sigma <_1 \tau$ means that σ lies below τ in the morass tree, with associated morass map $\pi_{\sigma\tau}$; $\sigma <_0 \tau$ means that σ and τ lie on the same morass level and σ is a smaller ordinal than τ.)

To obtain this morass, start with the morass constructed in [7], thinning out $S_{\omega_1^L}$ to consist of σ such that $T \restriction \sigma$ is uniformly $\Delta_1(L_\sigma)$ in parameter ω_1^L. Then define T_σ for $\alpha(\sigma) < \omega_1^L$ to be the tree on σ defined over L_σ with parameter $\omega_1^{L_\sigma}$ in the same way as T is defined over $L_{\omega_2^L}$ with parameter ω_1^L.

For $\alpha < \omega_1^L$ let $\sigma(\alpha)$ be the maximum of S_α. By induction on $\alpha < \omega_1^L$ we define $X_\alpha \subseteq \sigma(\alpha)$. For any $\sigma \in S_\alpha$ and $i < \alpha$, $\sigma(i)$ denotes the unique $\bar\sigma <_1 \sigma$ such that $\bar\sigma \in S_i$, if it exists. Now let $\langle \beta_i \mid i < \alpha \rangle$ be the least α-sequence in $L_{\sigma(\alpha)}$ of pairwise $T_{\sigma(\alpha)}$-compatible elements of $\sigma(\alpha)$ such that if $\sigma_i = \beta_i$ -th element of S_α then $\{i < \alpha \mid \text{For all } j < i,\ \sigma_j(i) \in X_{\alpha(\sigma_j(i))}\} = \emptyset$. Then X_α consists of all $\sigma \in S_\alpha$ such that there is $\bar\sigma <_1 \sigma$, $\pi_{\bar\sigma\sigma}$ cofinal, $\bar\sigma \in X_{\alpha(\bar\sigma)}$ together with each σ_i. (Ignore the σ_i's if $\langle \beta_i \mid i < \alpha \rangle$ does not exist.)

Let $X = (\mathrm{cof}\ \omega_1)^L \cup \{\sigma \in S_{\omega_1^L} \mid \text{There exists } \bar\sigma <_1 \sigma,\ \pi_{\bar\sigma\sigma} \text{ cofinal},\ \bar\sigma \in X_{\alpha(\bar\sigma)}\}$. If X contains a cardinal-preserving CUB subset C then note that $\sigma, \tau \in \{\beta \mid \text{ordertype } (C \cap \beta) = \beta\} \cap (\mathrm{cof}\ \omega_1)^L \rightarrow \sigma, \tau$ are T-compatible; otherwise $\{i < \omega_1^L \mid \cup\mathrm{Range}\pi_{\sigma(i)\sigma} \in X \text{ and } \cup\mathrm{Range}\pi_{\tau(i)\tau} \in X\}$ is empty, by construction, in contradiction to the fact that $C \cap \sigma$, $C \cap \tau$ are CUB in σ, τ of L-cofinality ω_1^L. Therefore T has a cardinal-preserving branch. Conversely, suppose that T has a cardinal-preserving, stationary, $\mathcal{P}(\omega_1^L)$-preserving branch b. For $\sigma \in S_{\omega_1^L}$, σ of L-cofinality ω_1^L and $i < \omega_1^L$, let σ^i denote $\cup\mathrm{Range}\pi_{\sigma(i)\sigma}$ (when $\sigma(i)$ is defined). By construction of X, if we let $X(\sigma)$ denote $\{i \mid \sigma^i \in X\}$ then $\{X(\sigma) \mid \sigma \in b^*\}$ generates a normal filter $\mathcal{F}$, where $b^* = \{\sigma \in b \mid \sigma = \text{ordertype } (S_{\omega_1^L} \cap \sigma)\} \cap (\mathrm{cof}\ \omega_1)^L$. But now there is a cardinal-preserving forcing (over $L[b]$) to add a CUB subset of X, using conditions which are closed, bounded subsets of X, making use of the fact that the $X(\sigma)$ for $\sigma \in b^*$ generate a normal filter to prove ω_2^L-distributivity.

Thus if T belongs to $\mathcal{T}^*(\omega_2^L)$ then X has a cardinal-preserving CUB subset, and if T belongs to $\sim \mathcal{T}^{**}(\omega_2^L)$ then X does not have a cardinal-preserving CUB subset. This proves the Lemma.

Lemma 5 immediately implies: $0^\# \longrightarrow_L \mathcal{C}(\omega_2^L)$. Now suppose that κ is an arbitrary L-regular cardinal greater than ω_2^L. We claim: $\mathcal{C}(\omega_2^L) \longrightarrow_L \mathcal{C}(\kappa)$. Indeed, let $\langle C_\alpha \mid \alpha \text{ singular in } L \rangle$ be an L-definable $\square$-sequence; i.e., C_α is cofinal in α of ordertype less than α and if $\bar\alpha$ is a limit point of C_α then

$C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ (see [7]). Given a constructible subset X of ω_2^L define $X^* \subseteq \kappa$ as follows: For $\alpha < \kappa$ of L-cofinality at most ω_1^L define $\alpha_0 = \alpha$, $\alpha_1 =$ ordertype of C_{α_0}, $\alpha_2 =$ ordertype of C_{α_1}, ... until you first reach $\alpha_k = \alpha^*$ less than ω_2^L. Then α belongs to X^* iff this α^* belongs to X. In addition X^* contains all ordinals less than κ of L-cofinality greater than ω_1^L. If X^* contains the cardinal-preserving CUB subset C^*, then X contains a cardinal-preserving CUB subset C, obtained by choosing β to be the ω_2^L-th element of C^* greater than ω_2^L and letting C consist of those α^* obtained from α in a final segment of $C^* \cap C_\beta$. And conversely, if X contains a cardinal-preserving CUB subset C then so does X^*, obtained by forcing over $L[C]$ with bounded, closed subsets of X^*. It follows that X contains a cardinal-preserving CUB subset iff X^* does, and therefore $0^\# \longrightarrow_L \mathcal{C}(\omega_2^L) \longrightarrow_L \mathcal{C}(\kappa)$. This completes the proof of (b).

(c) First note that in the proof of (b) we in fact showed

$$0^\# \longrightarrow_L (\mathcal{C}^*(\omega_2^L), \sim \mathcal{C}^{**}(\omega_2^L)),$$

where $\mathcal{C}^*(\omega_2^L) = \{X \subseteq \omega_2^L \mid X$ is Δ_1-definable over $L_{\omega_2^L}$ from parameter ω_1^L and contains a cardinal-preserving, $\mathcal{P}(\omega_1^L)$-preserving CUB subset$\}$, $\mathcal{C}^{**}(\omega_2^L) = \{X \subseteq \omega_2^L \mid X$ is Δ_1-definable over $L_{\omega_2^L}$ from parameter ω_1^L and contains a cardinal-preserving CUB subset$\}$. Now define

$\mathcal{A}(\omega_1^L) = \{Y \mid Y \in L,\ Y \subseteq \omega_1^L$ and $Y = \Lambda(R) \cap \omega_1^L$ for some cardinal-preserving real $R\}$,

where $\Lambda(R) =$ the class of R-admissible ordinals.

Lemma 6 $(\mathcal{C}^*(\omega_2^L), \sim \mathcal{C}^{**}(\omega_2^L)) \longrightarrow_L \mathcal{A}(\omega_1^L)$.

Proof of Lemma 6. Let X be a subset of the interval (ω_1^L, ω_2^L) which is Δ_1-definable over $L_{\omega_2^L}$ from the parameter ω_1^L. As in the proof of Lemma 5, we may choose a constructible morass such that to each $\sigma \in \cup\{S_\alpha \mid \alpha \leq \omega_1^L\}$ is assigned $X_\sigma \subseteq \sigma$ such that:

(1) For $\sigma \in S_{\omega_1^L}$, $X_\sigma = X \cap \sigma$.
(2) $\sigma <_0 \tau \rightarrow X_\sigma = X_\tau \cap \sigma$, $\sigma <_1 \tau \rightarrow X_\sigma = \pi_{\sigma\tau}^{-1}[X_\tau]$.

We set $Y = \{\eta \mid$ For some countable morass point σ, $\eta \in [\alpha(\sigma), \sigma^*]$ is X_{σ^*}-admissible and belongs to $X_{\sigma^*} \cup \{\alpha(\sigma), \sigma^*\}$, where $\sigma^* = \max S_{\alpha(\sigma)}\}$. Now consider the forcing P whose conditions are of the form (s, s^*, t) where:

(i) $s : |s| \to 2$, $|s| < \omega_1^L$ and $\eta \leq |s| \to$ (η is $s \restriction \eta$-admissible iff η belongs to Y).
(ii) $t : |t| \to 2$, $|t| < \omega_2^L$ and $\omega_1^L \leq \eta \leq |t| \to$ (η is $t \restriction \eta$-admissible iff η is X-admissible and belongs to $X \cup \{\omega_1^L\}$).
(iii) Let $P_{\omega_1^L}$ consist of t as in (ii) and let $G_{\omega_1^L}$ denote a $P_{\omega_1^L}$-generic. Then s^* is restraint for coding $G_{\omega_1^L}$ using conditions s as in (i).

Thus P factors as $P_{\omega_1^L} * P_\omega^{G_{\omega_1^L}}$ where $P_\omega^{G_{\omega_1^L}}$ codes the generic $G_{\omega_1^L}$ using conditions s as in (i) together with the usual restraints from almost-disjoint coding. $P_{\omega_1^L} \Vdash P_\omega^{G_{\omega_1^L}}$ is ω_1^L-distributive and ω_2^L-cc. Thus P is ω_2^L-preserving if $P_{\omega_1^L}$ is ω_2^L-distributive. If X contains a $\mathcal{P}(\omega_1^L)$-preserving CUB subset then the ω_2^L-distributivity of $P_{\omega_1^L}$ holds in such a $\mathcal{P}(\omega_1^L)$-preserving extension, and hence in L. And the $P_\omega^{G_{\omega_1^L}}$-generic $G_\omega \subseteq \omega_1^L$ can be coded by a cardinal-preserving real preserving admissibles, proving that Y belongs to $\mathcal{A}(\omega_1^L)$. Conversely, if $\Lambda(R) \cap \omega_1^L = Y$ for some cardinal-preserving real R then X contains the cardinal-preserving CUB subset of ω_2^L consisting of all $\alpha < \omega_2^L$ such that $L_\alpha[R]$ is Σ_1-elementary in $L_{\omega_2^L}[R]$. This proves the Lemma, and therefore the reduction $0^\# \longrightarrow_L \mathcal{A}(\omega_1^L)$.

Now notice that in the above we produced a constructible sequence $\langle Y_n \mid n \in \omega\rangle$ such that n belongs to $0^\#$ iff $Y_n = \Lambda(R) \cap \omega_1^L$ for some cardinal-preserving real R, and in addition the sets Y_n are Δ_1 over $L_{\omega_2^L}$ from the parameter ω_1^L. Now for any uncountable L-cardinal κ define Y_n^κ over L_κ in the same way that Y_n is defined over $L_{\omega_2^L}$. Then n belongs to $0^\#$ iff $Y_n^\kappa = \Lambda(R) \cap \kappa$ for some cardinal-preserving real R. The reduction $0^\# \longrightarrow \mathcal{A}(\kappa)$ therefore holds for every uncountable L-cardinal κ. This proves (c).

(d) Let $\mathcal{D}(\kappa)$ denote the set described in the statement of (d). We show: $(\mathcal{C}^*(\omega_2^L), \sim \mathcal{C}^{**}(\omega_2^L)) \longrightarrow_L \mathcal{D}(\omega_3^L)$. Suppose that X is a subset of the interval (ω_1^L, ω_2^L) which is Δ_1-definable over $L_{\omega_2^L}$ from parameter ω_1^L. If X contains a cardinal-preserving, $\mathcal{P}(\omega_1^L)$-preserving CUB subset C then in $L[C]$ the set

$X \cup \{\omega_2^L\}$ obeys the Condensation Condition at ω_1^L (see [5], Section 8.1). And in L we may define $Y \subseteq \omega_1^L$ such that $Y \cup \{\omega_1^L\} \cup X \cup \{\omega_2^L\}$ obeys the Condensation Condition at ω in L, and hence in $L[C]$. Thus by the Δ_1-Coding Theorem (Section 8.1 of [5]), $Y \cup \{\omega_1^L\} \cup X \cup \{\omega_2^L\}$ belongs to $\mathcal{D}(\omega_3^L)$. Conversely, if $Y \cup \{\omega_1^L\} \cup X \cup \{\omega_2^L\}$ belongs to $\mathcal{D}(\omega_3^L)$ for some $Y \subseteq \omega_1^L$ then X contains a cardinal-preserving CUB subset. Thus we have the desired reduction. As in the last part of the proof of (c), we also obtain the reduction $0^\# \longrightarrow_L \mathcal{D}(\kappa)$ for arbitrary L-cardinals $\kappa > \omega_2^L$. This completes the proof of Theorem 3. □

Proof of Theorem 4. This follows from the Condensation Condition in Section 8.1 of [5]. Suppose that A is a constructible subset of ω_2^L. We say that an extensional set x of ordinals *preserves* A if $\langle \bar{x}, \in, A \cap \bar{x} \rangle$ is isomorphic to $\langle x, \in, A \cap x \rangle$, where $\bar{x}$ denotes the transitive collapse of x. Suppose that A satisfies the condition

$(*)$ In L, the set of countable subsets of ω_2 which preserve A is stationary.

Then A obeys the Condensation Condition at ω and therefore by the Δ_1-Coding Theorem (Theorem 8.3 of [5]), A is the intersection with ω_2^L of a class which is Δ_1-definable over $L[R]$ without parameters, for some cardinal-preserving real R. Thus A belongs to $\mathcal{D}(\omega_2^L)$. Conversely, $(*)$ implies the Condensation Condition at ω and therefore the latter provides a constructible criterion for a set A to belong to $\mathcal{D}(\omega_2^L)$. (This argument is special to ω_2^L, as at cardinals greater than ω_2^L, $(*)$ is strictly weaker than the Condensation Condition.) □

Some Open Questions

1. Is there a reduction $\mathcal{T}(\omega_2^L) \longrightarrow_L \mathcal{C}(\omega_2^L)$? If so, the arguments of this paper could perhaps be simplified.
2. What happens if we weaken cardinal-preservation to the preservation of only some cardinals? For example, is the set
$\mathcal{C}'(\omega_3^L) = \{X \in L \mid X \subseteq \omega_3^L$ and X has a CUB subset in an extension of L which preserves $\omega_1^L, \omega_3^L\}$
constructible?
3. What is the situation with the tree problem at κ, when trees on κ are required to have levels of size less than κ?

References

[1] Baumgartner J., Harrington L. and Kleinberg E. *Adding a closed unbounded set*, Journal of Symbolic Logic, Vol. 41, pp. 481–482, 1976.

[2] Devlin, K. and Jensen, R., *Marginalia to a theorem of Silver*, Springer Lecture Notes 499, pp. 115–142, 1975.

[3] Friedman S. *The Π^1_2-singleton conjecture*, Journal of the AMS, Vol. 3, pp. 771–791, 1990.

[4] Friedman, S. *Generic saturation*, Journal of Symbolic Logic, Vol. 63, pp. 158–162, 1998.

[5] Friedman, S. **Fine Structure and Class Forcing**, de Gruyter Series in Logic and its Applications, 2000.

[6] Harrington L. and Kechris A. *Π^1_2-singletons and $0^{\#}$*, Fundamenta Mathematicae 95, 1977, no. 3, 167–171.

[7] Jensen R. *The fine structure of the constructible hierarchy*, Annals of Mathematical Logic, 1972.

[8] Martin D. A. and Solovay R. M. *A Basis Theorem for Σ^1_3 Sets of Reals*, Annals of Mathematics, 1969.

[9] Stanley M. *Forcing closed unbounded subsets of ω_2*, to appear.