Capturing the Universe

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Abstract

We describe the universe V of sets using the ideas of *mouse iteration* from large cardinal theory and *forcing* from the theory of independence. We introduce *Mighty Mouse*, an absolute mouse of modest strength and show that it *captures* V in the sense that V is a class-generic extension of a definable inner model resulting from a definable iteration of Mighty Mouse via a forcing that is definable and whose antichains are sets.

A key tool is the *Stable Core* of [7]. We show that Mighty Mouse *strongly captures* the Stable Core in the sense that the Stable Core is definable over a definable inner model resulting from a definable iteration of Mighty Mouse. Applying the methods used to prove this result, we characterise the reals of the *Enriched Stable Core* of [8] (assuming Ord is Mahlo) and show that the Stable Core is not rigid and therefore V is class-generic over a definable, non-rigid inner model (assuming the existence of a satisfaction predicate for V).

Introduction

The universe of sets V is important for the foundations of mathematics as it provides an arena in which virtually all mathematical constructions can be carried out. But what does V look like? Can it be described using the tools that set-theorists have for building universes of set theory?

Gödel [13] provided us with the universe L of constructible sets, an important inner model (subuniverse) of V with remarkable combinatorial properties and clear internal structure ([14]). Cohen [3] later produced a method for creating new universes from old, the forcing method, which can be used to obtain *generic extensions* of L which are larger than L. Is V simply a generic extension of L? If so, then V can be described using just the methods of constructibility and forcing.

However further work of Scott [19] and Silver [21] revealed that V cannot be a generic extension of L if *large cardinals* exist. Large cardinals are essential to set theory as they are needed to show that important set-theoretic phenomena are consistent with the traditional axioms for set theory.

So to achieve our goal of describing V we need something more, and this is the notion of *mouse* (first introduced in [4]). To explain mice we take a closer look at the type of large cardinal that Scott considered, a *measurable* cardinal. Let us say that U is a *measure* on a set X if U is a collection of subsets of X such that for any subset Y of X, either Y or $X \setminus Y$ belongs to U and whenever U_0 is a subcollection of U of size less than the size of X, the intersection of the sets in U_0 belongs to U. We say that U is *nonprincipal* if U consists only of infinite sets and we say that X is *measurable* if X is uncountable and there is a nonprincipal measure on X. A cardinal number[1] κ is *measurable* if it is the cardinality of a measurable set, which can be taken to be κ itself. Scott's Theorem states that if there is a measurable cardinal, then V is larger than L.

If U is a nonprincipal measure on the uncountable cardinal κ then we can form the *universe of sets constructible from* U, denoted by $L[U]$. Silver [20] showed that $L[U]$ is a very nice "L-like" model, sharing many of the properties of Gödel's L. Like L, $L[U]$ is not a set, but a *class*, as it contains all ordinal numbers. However in his analysis of $L[U]$, Silver was led to study smaller versions $m = \bar{L}[\bar{U}]$ of $L[U]$ which are sets and in which $\bar{U}$ has the appearance of a measure in m. Further deep work of Dodd and Jensen [4] took the first key steps in developing this idea into a theory of what we now call *mice*.

[1]Using the axiom of choice, every set X can be put into 1-1 correspondence with an ordinal number and the least such ordinal number is called the *cardinality* of X. The *cardinal numbers* are simply the cardinalities of the sets. We assume the axiom of choice throughout.

Not every set-sized version $\bar{L}[\bar{U}]$ of $L[U]$ is a mouse; like $L[U]$ a mouse must not only be well-founded[2] but must also be *iterable*. The latter means the following. If $m = \bar{L}[\bar{U}]$ is a set-sized version of $L[U]$, then there is a natural way to form its *ultrapower* $\mathrm{Ult}(m)$ with a natural embedding from m into $\mathrm{Ult}(m)$. For m to be *iterable* both m and $\mathrm{Ult}(m)$ must be well-founded. And we can repeat this, forming $\mathrm{Ult}(\mathrm{Ult}(m))$; this too must be well-founded. In fact, we require that well-foundedness is not lost when we continue this process of forming iterated ultrapowers for any ordinal number of stages, taking direct limits at limit stages. Without the iterability requirement we say that m is a *premouse*; a *mouse* is an iterable premouse. Iterability helps ensure uniqueness; for example there is a unique mouse called $0^{\#}$ which is contained in all other mice. And this least mouse is not generic over L (a strong form of Scott's Theorem).

So far we have only discussed mice which are set-sized versions of $L[U]$, a model with just one measurable cardinal. But it is not difficult to generalise the above discussion to models with more measurable cardinals. A particularly interesting case is a model with a measurable limit of measurable cardinals; the analogue of $0^{\#}$ for this property is denoted here by $m_1^{\#}$. Just as $0^{\#}$ is *the least mouse with a measurable cardinal*, $m_1^{\#}$ is the least mouse with a measurable limit of measurable cardinals [3].

The reason for introducing $m_1^{\#}$ goes beyond mere generalisation. This mouse can be used to *strongly capture* something significant about the set-theoretic universe. Let Card denote the class of all cardinal numbers. In analogy to the enlargement of Gödel's L to the universe $L[U]$ of sets constructible from the measure U, we can also form the universe $L[\mathrm{Card}]$ of sets constructible from the class Card. Then $m_1^{\#}$ *strongly captures* the universe $L[\mathrm{Card}]$ in the following sense:

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[2]This means that there are no infinite decreasing sequences through its ordinals, which is the case for V and its subuniverses.

[3]An important technical point: In $L[U]$ and its generalisations to models built from sequences of measures, each measurable cardinal carries a measure which is an element of the model. This will fail for the mice $0^{\#}$ and $m_1^{\#}$, which each have a largest measurable κ whose measure is not an element of the mouse but instead a *predicate* consisting of the subsets of κ of the mouse which belong to the measure. So for example $0^{\#}$ is not simply a transitive set, but is a transitive structure of the form $(\bar{L}[\bar{U}], \bar{U})$ which is *amenable*, i.e., the intersection of $\bar{U}$ with each element of $\bar{L}[\bar{U}]$ is also an element of $\bar{L}[\bar{U}]$.

Theorem 1 *$L[Card]$ is definable over the truncation at Ord of an iterate of $m_1^\#$ resulting from a definable iteration of length Ord.*

We explain this result as follows. Recall that $m_1^\#$ is the least mouse with a measurable limit of measurable cardinals. For simplicity of notation let m denote $m_1^\#$. Now form the ultrapower $\mathrm{Ult}(m)$ using the measure on the least measurable cardinal of m. Then form the ultrapower $\mathrm{Ult}^2(m)$ of $\mathrm{Ult}(m)$ using the measure on the least measurable cardinal of $\mathrm{Ult}(m)$. Iterate in this way for $\aleph_1$ steps, where $\aleph_1$ denotes the least uncountable cardinal (of V). Then $\aleph_1$ is a measurable cardinal in this iterate. Use the 2nd measure of this iterate for $\aleph_2$ steps, where $\aleph_2$ denotes the 2nd uncountable cardinal. Then $\aleph_1$ and $\aleph_2$ are the first two measurable cardinals of this new iterate. Keep iterating until $\aleph_n$ is the n-th measurable cardinal of the resulting iterate for each finite n. Then iterate the next available measurable cardinal up to $\aleph_{\omega+1}$, the least cardinal greater than the supremum of the $\aleph_n$'s. Continue this until one reaches an iterate m^* where all measurables below the largest measurable are successor cardinals of V; as the iteration is definable the replacement scheme ensures that such an iterate will be reached. Then take an ultrapower of m^* using the measure on its largest measurable κ; as κ is a limit of measurables this will create new measurables above κ and the iteration may continue, moving measurable cardinals onto successor V-cardinals. After Ord steps, one reaches a class-sized iterate m_∞, taller than Ord, where Ord is the largest measurable and the smaller measurables are exactly the uncountable successor V-cardinals. Thus if we truncate m_∞ at Ord, keeping only its information strictly below Ord, we obtain a structure $m_\infty|Ord$ whose measurables are exactly the successor V-cardinals. Therefore the class Card (the closure of the class of successor V-cardinals) is definable over $m_\infty|\mathrm{Ord}$.

In general:

Definition 2 *(Tentative definition[4]) A mouse m* strongly captures *an inner model M if M is definable over the truncation at Ord of an iterate of m resulting from a definable iteration of length Ord[5].*

[4]When considering non-linear iterations, via iteration trees, this definition must be slightly modified; see the end of the next section.

[5]In this definition, by "definable" we mean V-definable with parameters.

Thus $m_1^\#$ strongly captures L[Card]. Using stronger mice we can strongly capture larger models. For example, if we use a mouse with a measure U on a cardinal κ such that the set of measurable cardinals less than κ belongs to U (a measure of *positive Mitchell order*) then we can strongly capture L[Reg], where Reg denotes the class of regular cardinals.[6] We can also strongly capture L[Cof], where Cof denotes the cofinality function.[7] For this we require a mouse with a measurable κ *of Mitchell order* $\kappa + 1$[8]. (For these results about L[Reg] and L[Cof], see [9]. In general, the mentioned mice are required for these strong capturing results.)

Can a mouse strongly capture the entire universe V? No, because no mouse can be an element of any of its iterates. However, recall that we have a second method for building universes of set theory, the forcing method.

Definition 3 *A mouse m* captures *an inner model M if M is a generic extension via a definable forcing*[9] *of an inner model strongly captured by m.*

Our main result is the following:

Theorem 4 *V is a generic extension via a definable forcing of a definable inner model which is strongly captured by some mouse; moreover, the forcing is Ord-cc and the mouse is* absolute *in the sense that it has a fixed definition in any inner model that contains it*[10].

The least mouse witnessing Theorem 4 is called *Mighty Mouse*, denoted mm. Theorem 4 is proved by showing that mm strongly captures (a version of) the *Stable Core* of [7]. Then we apply the main result of [7], asserting that

[6] A cardinal κ is regular if the union of fewer than κ sets of size less than κ still has size less than κ. A successor cardinal (i.e., a successor in the increasing enumeration of cardinals) is regular, but a limit cardinal need not be regular.

[7] The *cofinality* of an infinite cardinal κ is the least cardinal μ such that κ is the union of a size μ family of sets, each of size less than κ.

[8] We don't define Mitchell order here, but the idea is that a measure has Mitchell order at least 1 if it "concentrates" on measurables, it has Mitchell order at least 2 if it "concentrates" on measurables which carry measures of Mitchell order at least 1, etc.

[9] In this context, N is a "generic" extension of M if $N = M[G]$ where for some M-definable, ZFC-preserving class forcing P, G is P-generic with respect to M-definable dense classes.

[10] In fact it is coded by a Π^1_2-singleton.

V is a generic extension of the Stable Core (via a definable Ord-cc forcing, i.e., a definable forcing all of whose definable antichains are sets).

Thus we have achieved our initial goal: V can be described through the methods of mouse iteration and forcing.

Theorem 4 has some additional consequences (also see Section 4):

1. *The Generic IMH.* The *IMH* (Inner Model Hypothesis) was introduced in [6]. In its simplest version, it asserts that if a sentence holds in an outer model of V then it holds in an inner model of V. The *Generic IMH* is the weaker statement that if a sentence holds in an outer model of V then it holds in a generic extension of an inner model of V. Theorem 4 implies that the Generic IMH does hold if Mighty Mouse = mm exists. For, if φ holds in an outer model W of V, then using the absoluteness of mm, W is a definably-generic extension of a definable inner model which is strongly captured by mm. But it is easily shown that this definable inner model can be taken to be elementarily equivalent to K_{mm}, the truncation at Ord of the iterate of mm obtained by simply iterating its top measure Ord-many times; thus K_{mm} is an inner model of V and φ holds in a generic extension of K_{mm}. If we further assume that for each set x, mm_x, Mighty Mouse relativised to x, exists then the Generic IMH with arbitrary set-parameters holds in V.

2. *A unique Multiverse.*[11] Let $\mathbb{M}(V)$ denote the multiverse obtained from V by closing $\{V\}$ under generic extensions, elementary embeddings and their inverses. That is, $\mathbb{M}(V)$ is the smallest multiverse satisfying: V is in $\mathbb{M}(V)$ and if V_0 is a forcing extension of V_1 or elementarily embeds into V_1 (with the same ordinals as V_0) and either V_0 or V_1 is in $\mathbb{M}(V)$ then both V_0 and V_1 are[12]. Then if V_0 and V_1 both contain Mighty Mouse it follows that $\mathbb{M}(V_0)$ equals $\mathbb{M}(V_1)$. This is because both V_0 and V_1 are generic extensions of universes into which K_{mm} embeds.

[11] There are many notions of "Multiverse" (collection of universes) in set theory. For a general discussion, see [1]. Even in the simplest case, that of the "set-generic multiverse", the question arises: from where are the elements of the multiverse to be taken? The clearest answer is to treat the initial universe V as a countable transitive model of ZFC and take the elements of the multiverse from the background universe in which V lives.

[12] As in the previous footnote, these forcings and elementary embeddings can be taken from the background universe in which V, regarded as a countable transitive model, lives.

3. *The Stable Core is small.* In [7] I asked if the Stable Core is a good approximation to V in the senses that weak covering holds relative to it, large cardinals are witnessed by it, it is rigid and V is generic over it. This question is answered by the results of the present paper: By Theorem 4 the first two of these properties fail. Rigidity is shown to fail in Section 4. The genericity of V over the Stable Core was already established in [7].

4. *Maximality.* In [12] we argued that the maximality of the universe *in height* is captured by the notion of #-generation. The Generic IMH with parameters is a strong statement of maximality in width. Thus by Theorem 4, a consistent and appealing maximality principle for V in both height and width is captured by the existence of mm_x for each set x together with #-generation, whose consistency strength is far below one Woodin cardinal.

This paper is structured as follows. In Section 1 we discuss the theory of mice, introducing the mouse mm. In Section 2 we present a version of the Stable Core well-suited to our purposes and show that V is a generic extension of it. In Section 3 we show that Mighty Mouse strongly captures the Stable Core, finishing the proof of Theorem 4. In Section 4 we discuss corollaries and variants of Theorem 4, and prove that the use of Mighty Mouse is optimal for our results.

Woodin's Extender Algebra and Genericity Iterations These methods (see [23]) can be used, without the techniques of the present paper, to obtain a weak version of our main result. Woodin showed that if x is a set of ordinals and m is a mouse with a Woodin cardinal then x is generic over an iterate of m. A similar method can be used to show:

(Woodin) There is an iteration of Mighty Mouse of length Ord with common part model $M = L[\vec{E}]$ such that V is Ord-cc generic over $(M, \vec{E})$.

The present paper improves on this in several ways:

(a) Our iteration is *definable.*

Woodin's iteration is definable relative to a class of ordinals A such that $V = L[A]$. Our iteration is definable relative to a definable refinement of the Stability Predicate and is therefore definable.

(b) The iteration tree resulting from our iteration has a (unique) branch of length Ord (assuming the existence of a satisfaction predicate for V) and therefore our common part model is the truncation at Ord of a wellfounded Ord-iterate of Mighty Mouse.

(c) V is generic over our common part model $(M, \vec{E})$ via an Ord-cc forcing that is definable over M, not just definable over $(M, \vec{E})$.

Improvements (a) and (b) make heavy use of the methods of this paper. Improvement (c) can also be obtained with Woodin's iteration, applying deep work of Steel [22] to argue that $\vec{E}$ is definable over M.

Most importantly, Woodin's iterations do not give strong capturing and in particular do not suffice to show that the Stable Core is definable over the common part model of an iteration of any mouse. Our iteration in this paper expresses stability in V in terms of strength in a truncated Ord-iterate M of Mighty Mouse, showing that the Stability Predicate is not only generic, but in fact definable over M.

1. Mice

Mice serve as approximations to models with large cardinals, and large cardinals are often best formulated using the notion of *elementary embedding*. In partciular, this is the case for the large cardinal notions central to this paper. A function[13] $j : M \to N$ from from one model of set theory to another is an *elementary embedding* if for any $x_1, \ldots, x_n$ in M, a first-order property[14] φ is true for $x_1, \ldots, x_n$ in M exactly if it is true for $j(x_1), \ldots, j(x_n)$ in N. In particular $x_1 \in x_2$ iff $j(x_1) \in j(x_2)$.

Theorem 5 *(Scott [19]) The following are equivalent.*
(a) κ is a measurable cardinal.
(b) There is an elementary embedding $j : V \to M$ with critical point *κ, i.e., such that $j(\alpha) = \alpha$ for ordinals $\alpha < \kappa$ and $j(\kappa) > \kappa$.*

[13]We make free use in this paper of functions whose domains are proper classes. This is easily formalised using the standard system GB, Gödel-Bernays class theory.

[14]A first-order property is a property that is expressible using the membership relation $\in$, equality $=$, variables, logical connectives like *and*, *or* and *not* and quantifiers *for all* and *there exists* that range over elements of the universe.

We can strengthen measurability by imposing further requirements on M in the emedding $j : V \to M$ witnessing the measurability of its critical point κ. By requiring M to contain $H(\alpha)$ = the union of all transitive sets of size less than α for a cardinal α (perhaps depending on the embedding $j : V \to M$) which is larger than κ^+ = the least cardinal greater than κ, we express the idea of M being "close" to V.

Definition 6 *A cardinal κ is* 1-strong *(or just* strong*) if for any cardinal α there is $j : V \to M$ with critical point κ such that $j(\kappa)$ is greater than α and M contains $H(\alpha)$. It is* $n+1$-strong *(for finite $n > 0$) if we also require that V and M have the same n-strongs less than α.*

If κ is a measurable cardinal then $H(\kappa)$ is a model of set theory and we can talk about the existence of large cardinals in this "local" universe $H(\kappa)$. We are interested in a universe in which there is a measurable cardinal κ such that for each n, the cardinals less than κ which are n-strong in the truncated universe $H(\kappa)$ are unbounded in κ. Just as in the case of a measurable cardinal (without the requirement of n-strongs in $H(\kappa)$) we can discuss premice which are set-sized versions of a universe with a measurable κ and unboundedly many n-strongs in $H(\kappa)$ for each n. There is also a corresponding notion of *iteration* and *iterability* as well as a least iterable premouse (i.e., mouse) of this type. This is *Mighty Mouse*, denoted mm.

To help explain iterations of mm we sketch the theory of iteration trees (due to Martin and Steel [15]). First we make some remarks about $L[\vec{E}]$-*models*. These are models obtained by relativising Gödel's constructible universe to a predicate $\vec{E}$ built from *extenders*. Just as L is the union of a hierarchy $(L_\alpha \mid \alpha \in \mathrm{Ord})$ where L_α consists of the first α levels of L, so is $L[\vec{E}]$ the union of a hierarchy $(L_\alpha[\vec{E}] \mid \alpha \in \mathrm{Ord})$. Here, $\vec{E} = (E_\alpha \mid \alpha \in \mathrm{Ord})$ is a sequence of *extenders*, which means that either E_α is empty or is a cofinal elementary embedding $E_\alpha : L_{\bar{\alpha}}[\vec{E}] \to L_\alpha[\vec{E}]$ where $\bar{\alpha}$ is the least cardinal of $L_\alpha[\vec{E}]$ greater than the critical point of E_α. Further conditions are imposed to ensure that $L[\vec{E}]$ shares many of the nice features of L. In addition, an extender E_α can be used to form an *ultrapower* $\mathrm{Ult}(L_\beta[\vec{E}], E_\alpha)$ into which $L_\beta[\vec{E}]$ naturally embeds, assuming $\beta \geq \alpha$ and $\bar{\alpha}$ is still a cardinal in $L_\beta[\vec{E}]$. The $\vec{E}$-sequences of $L_\beta[\vec{E}]$ and $\mathrm{Ult}(L_\beta[\vec{E}], E_\alpha)$ agree below α and if $\beta > \alpha$ then although α is not a cardinal of $L_\beta[\vec{E}]$, it is a cardinal in $\mathrm{Ult}(L_\beta[\vec{E}], E_\alpha)$.

We iterate $L[\vec{E}]$ by successively applying extenders to form ultrapowers, just as in the case of $L[U]$, with one important difference: At each stage we choose an extender from the model we have reached in the iteration but instead of applying it to that same model (as above), we have the freedom to apply it to a model that appeared earlier in the iteration. This sequence of iterates of the initial $L[\vec{E}]$ model carries the structure of a tree, known as an *iteration-tree*: At stage α, if we form $\mathrm{Ult}(M_i, E)$ where E is taken from the α-th model M_α and M_i denotes the model with index $i \leq \alpha$, then we place $\alpha + 1$ as an immediate successor of i in the tree. In the iterations that we will consider in this paper, i will always be chosen as the least i for which we can form $\mathrm{Ult}(M_i, E)$ (i.e., so that the least cardinal greater than the critical point of E is the same in M_i as it is in M_α). Moreover, in our iteration of Mighty Mouse in Section 3, there will be only one branch through the iteration tree cofinal in λ at limit stages λ. *Iterability* implies that this unique cofinal branch is well-founded, i.e., is such that the direct limit of the models indexed along the branch is wellfounded. An ordinal is a predecessor of λ in the tree order if it is a predecessor of one of the elements of the unique branch cofinal in λ [15].

Everything said in the previous paragraph about $L[\vec{E}]$ also applies to *premice* which are just set-sized versions of $L[\vec{E}]$ models. An iterable premouse is a *mouse*.

Definition 7 *Mighty Mouse is the least mouse $L_\alpha[\vec{E}]$ with a measurable cardinal*[16] *κ that is a limit of κ, n-strongs, i.e., of cardinals which are n-strong in $L_\kappa[\vec{E}]$, for each n.*

The meaning of "least" in the above definition is that there is a Σ_1-elementary embedding of Mighty Mouse into an initial segment $L_{\bar{\beta}}[\vec{F}]$, $\bar{\beta} \leq \beta$ of any mouse $L_\beta[\vec{F}]$ with a measurable κ which is a limit of κ, n-strongs for

[15] Our iterations in Section 4 of larger mice give rise to iteration trees which may have more than one cofinal, well-founded branch at a limit stage. In this case *iterability* has a more subtle meaning: it says that there is an *iteration strategy*, i.e., a method for choosing cofinal well-founded branches at limit stages which ensures that no matter how extenders are chosen at successor stages, such cofinal well-founded branches are available at every limit stage (provided the strategy was used at earlier limit stages).

[16] As with the mice discussed earlier, the measure witnessing measurability (of the largest measurable) need not be an element of the mouse, but instead an amenable predicate.

each n. It follows from the existence of (much less than) a Woodin cardinal that mm exists.

We end this section with a clarification of *strong capturing*. With linear iterations (i.e., where at each stage γ the chosen exttender E_γ is applied to m_γ and not to some $m_{\bar{\gamma}}$, $\bar{\gamma} < \gamma$), we say that m *strongly captures* an inner model M if M is definable over the truncation at Ord of an iterate of m resulting from a definable iteration of length Ord. But when using non-linear iterations we must modify this definition, as a definable iteration tree of length Ord may have no branch of length Ord and therefore no "Ord-iterate". Instead we talk about the *common part model* of the iteration.

Definition 8 *Suppose that* $(m_\alpha \mid \alpha < Ord)$ *is an iteration of a mouse along an iteration tree. The* common part model *of the iteration is the union of all* m *such that* m *is an initial segment of* m_α *for all sufficiently large* α*.*

In practice, iterations are formed using extenders E_γ with increasing index, and this implies that the common part model of the iteration is a well-defined *weasel* (mouse of height Ord). If the iteration tree does have a branch of length Ord, then the common part model is the truncation at Ord of the direct limit of the models along that branch.

Definition 9 *A mouse* m *strongly captures an inner model* M *if* M *is definable over the common part model of a definable Ord-iteration of* m*.*

In general, for an Ord-iteration $(m_\gamma \mid \gamma < \mathrm{Ord})$ to be definable we only require that the sequence $(m_\gamma \mid \gamma < \mathrm{Ord})$, the sequence of extenders used and the associated iteration tree be definable; we do not require that this tree has a definable cofinal branch. For example, if there is a satisfaction predicate for V, then the iteration tree resulting from the definable iteration we use to strongly capture the Stable Core has a unique branch of length Ord, this branch is wellfounded (i.e., the direct limit of the models indexed along this branch is wellfounded) but is not definable. However the common part model of any definable iteration of length Ord is definable.

2. The Stable Core

In this section we introduce a slight variant of the Stable Core of [7] and establish its basic properties, the most important of which is that V is one of its generic extensions. This slight change in the definition of the Stablity Predicate is needed for the proof in the next section that it is definable over an iterate of Mighty Mouse. Although this section is very close to what is already presented in [7], the definition of the Stability Predicate used here is slightly different[17] and therefore the arguments of [7] have to be redone in order to be convincing. Our treatment here of the Stable Core is self-contained, without any need to refer back to [7].

Recall that for an infinite cardinal α, $H(\alpha)$ is the union of all transitive sets of size less than α. A useful fact is that $H(\alpha)$ is a Σ_1-elementary substructure of V for uncountable cardinals α and therefore if $\alpha < \beta$ are both uncountable cardinals then $H(\alpha)$ is Σ_1-elementary in $H(\beta)$. Note that if β is a strong limit cardinal (i.e., an uncountable cardinal such that $2^\gamma < \beta$ for $\gamma < \beta$) then $H(\alpha)$ is an element of $H(\beta)$ for infinite cardinals $\alpha < \beta$ and $H(\beta)$ has cardinality β. In this case we write $\mathbb{H}(\beta)$ for the amenable[18] structure $(H(\beta), H \restriction \beta)$ where $H \restriction \beta = \{(\alpha, H(\alpha)) \mid \alpha \text{ is an infinite cardinal less than } \beta\}$.

We say that α is *β, n-stable* if $\alpha < \beta$ are strong limit cardinals and $\mathbb{H}(\alpha)$ is Σ_n-elementary in $\mathbb{H}(\beta)$. (For $n = 0$ the latter condition is automatic.) Using the fact that $H(\alpha)$ is Σ_1-elementary in $H(\beta)$ for uncountable cardinals $\alpha < \beta$, this can be seen to be equivalent to saying that $\alpha < \beta$ are strong limit cardinals and $H(\alpha)$ is Σ_{n+1}-elementary in $H(\beta)$.

Let $\mathrm{Stable}_n(\beta)$ denote the set of α which are β, n-stable. We say that α is *nicely β, n-stable* if α is β, n-stable and in addition, if $n > 0$, the $\beta, (n-1)$-stables are cofinal in β[19]. If the β, n-stables are cofinal in β then α is (nicely) $\beta, (n+1)$-stable iff $(\mathbb{H}(\alpha), \mathrm{Stable}_n(\alpha))$ is Σ_1-elementary in $(\mathbb{H}(\beta), \mathrm{Stable}_n(\beta))$

[17] As remarked in [8], the arguments of [7] require that the definition of the Stability Predicate be modified to incorporate the notion of "n-Admissility". It is not clear how to strongly capture the Stable Core if the Stability Predicate is defined using this notion and for this reason we redefine the Stability Predicate without it for the purposes of the present paper.

[18] A structure (T, A) with T transitive is *amenable* if $A \cap t$ belongs to T for each t in T. Amenability ensures the existence of a predicate which is universal for predicates which are Σ_n-definable over (T, A), for each n.

[19] Note that whether or not a β, n-stable α is nicely β, n-stable depends only on β and not on α.

[20]. Note that if α is $\beta, (n+1)$-stable then the α, n-stables are cofinal in α (and are equal to the β, n-stables less than α), but the β, n-stables need not be cofinal in β.

The *Stability Predicate* S consists of all triples (α, β, n) such that α is nicely β, n-stable. The predicate S is Δ_2 definable. The *Stable Core* is the structure $(L[S], S)$.

Theorem 10 *(In Gödel-Bernays class theory) V is generic over the Stable Core. More precisely, for some $(L[S], S)$-definable and Ord-cc forcing Q, there is a G which is Q-generic over $(L[S], S)$ such that $V = L[G]$ and $(L[G], G)$ is a model of ZFC. G is generic over V for a definable forcing and if there is a satisfaction predicate for V, G is definable over $(V, T, <)$ where T is the V-amenable predicate $\{(\alpha, n) \mid \alpha$ is Ord, n-stable, $n \in \omega\}$ and $<$ is a wellorder of V of length Ord. The same is true with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model M.*

The proof of Theorem 10 comes in two parts. First we show that V can be written as $L[F]$ where F is a function from the ordinals to 2 which is *neutral* for the Stability Predicate S in the sense that whenever α is nicely $\beta, (n+1)$-stable then α is also nicely $\beta, (n+1)$-stable "relative to F". Then we use this function F to prove the genericity of V over $(M[S], S)$ for any definable inner model M.

A stability-neutral function

Our aim is to produce a function F from the ordinals to 2 which codes V (i.e., which satisfies $V = L[F]$) and which *is neutral for the Stability Predicate*. The latter means that if $\alpha < \beta$ are strong limit and α is nicely β, n-stable then α is β, n-stable relative to F, i.e., $(\mathbb{H}(\alpha), F \restriction \alpha)$ is Σ_n-elementary in $(\mathbb{H}(\beta), F \restriction \beta)$[21]. (As $\beta, (n-1)$-stability implies $\beta, (n-2)$-stability for

[20]This explains the use of the word "nicely" in *nicely $\beta, (n+1)$-stable*: it ensures that $\beta, (n+1)$-stability can be expressed as $\beta, 1$-stability relative to the additional predicate consisting of the β, n-stables. Note that if α is β, n-stable then whether or not α is nicely β, n-stable depends only on β and not on α.

[21]Note that although $H(\alpha)$ is Σ_1-elementary in V for uncountable cardinals α, this may fail relative to F; for this reason we cannot assert that Σ_n-elementarity in $(\mathbb{H}(\beta), F \restriction \beta)$ coincides with Σ_{n+1}-elementarity in $(H(\beta), F \restriction \beta)$.

$n > 1$, it then follows that α is *nicely* β, n-stable relative to F, i.e., that the $\beta, (n-1)$-stables relative to F are cofinal in β.)

Let C denote the class of strong limit cardinals. We define by induction on $\beta \in C$ a collection $P(\beta)$ of functions from β to 2 with the property that if $\alpha < \beta$ are in C and p belongs to $P(\beta)$ then $p \restriction \alpha$ belongs to $P(\alpha)$.

If β is not a limit point of C then $P(\beta)$ consists of all functions $p : \beta \to 2$ such that $p \restriction \alpha$ belongs to $P(\alpha)$ for all $\alpha \in C \cap \beta$. (Such functions exist, assuming that $P(\alpha)$ is nonempty for all $\alpha \in C \cap \beta$, a fact that we will verify later.)

If β is a limit point of C then let $P(<\beta)$ be the union of the $P(\alpha)$ for α in $C \cap \beta$, ordered by extension. Assuming *extendibility for* $P(<\beta)$, i.e., the statement that for $\alpha_0 < \alpha_1 < \beta$ in C, each q_0 in $P(\alpha_0)$ can be extended to some q_1 in $P(\alpha_1)$, this forcing adds a generic function with domain β, which we denote by $\dot{f} : \beta \to 2$. For $n > 0$ we say that $p : \beta \to 2$ is *n-generic for* $P(<\beta)$ if $G(p) = \{p \restriction \alpha \mid \alpha \in C \cap \beta\}$ meets every dense subset of $P(<\beta)$ of the form $\{q \in P(<\beta) \mid q \Vdash \forall x \varphi \text{ or } q \Vdash \neg\varphi(\sigma) \text{ for some } P(<\beta)\text{-name } \sigma\}$, where φ is a $\Sigma_{n-1}(\mathbb{H}(\beta), \dot{f})$ sentence with parameters from $H(\beta)$. We define $P(\beta)$ to consist of all $p : \beta \to 2$ which are n-generic for $P(<\beta)$ for all n such that $\mathbb{H}(\beta)$ is n-admissible, i.e., satisfies Σ_n-replacement.

Let P be the union of all of the $P(\beta)$'s, ordered by extension.

Lemma 11 *Assume Extendibility for P. Suppose that G is P-generic over V (meeting V-definable dense classes) and let F be the union of the functions in G. Then $V = L[F]$ and F is neutral for the Stability Predicate. Moreover, V satisfies replacement with F as an additional predicate.*

Proof. Extendibility implies that it is dense to code any set of ordinals into the P-generic function F, from which it follows that V is contained in $L[F]$. As $F \restriction \alpha$ belongs to V for each $\alpha \in C$ it also follows that $L[F]$ is contained in V and therefore $L[F]$ equals V.

To see that F is neutral for the Stability Predicate, first note that if $\dot{f}$ denotes the $P(<\beta)$-generic function, then the relation $q \Vdash \gamma$ for q in $P(<\beta)$ and $\Pi_1(\mathbb{H}(\beta), \dot{f})$ sentences γ with parameters from $H(\beta)$ is Π_1 over $\mathbb{H}(\beta)$:

$q \Vdash \gamma$ iff for all $r \leq q$ and strong limit $\alpha \leq \mathrm{Dom}(r)$, $(\mathbb{H}(\alpha), r \restriction \alpha) \models \gamma$. It then follows by induction on $n \geq 1$ that the relation $q \Vdash \gamma$ for q in $P(<\beta)$ and $\Pi_n(\mathbb{H}(\beta), \dot{f})$ sentences γ with parameters from $H(\beta)$ is Π_n over $\mathbb{H}(\beta)$.

Now suppose that $n > 0$ and α is nicely β, n-stable. As $\mathbb{H}(\alpha)$ is n-admissible, $F \restriction \alpha$ is n-generic for $P(<\alpha)$. It follows that any $\Pi_n(\mathbb{H}(\alpha), \dot{f} \restriction \alpha)$ sentence γ with parameters from $H(\alpha)$ which is true in $(\mathbb{H}(\alpha), F \restriction \alpha)$ is forced in $P(<\alpha)$ by some condition $F \restriction \alpha_0$, $\alpha_0 < \alpha$. Then as α is β, n-stable, $F \restriction \alpha_0$ also forces γ in $P(<\beta)$. It follows that $F \restriction \alpha_0$ forces γ in $P(<\beta_0)$ for all $\beta, (n-1)$-stable β_0 greater than α. If $n = 1$ then γ holds in $(\mathbb{H}(\beta_0), F \restriction \beta_0)$ for such β_0 (else $F \restriction \alpha_0$ could not force γ in $P(<\beta_0)$) and therefore since such β_0's are cofinal in β, γ holds in $(\mathbb{H}(\beta), F \restriction \beta)$. If $n > 1$ then such β_0 are $(n-1)$-admissible and therefore $F \restriction \beta_0$ is $(n-1)$-generic. It follows that γ holds in $(\mathbb{H}(\beta_0), F \restriction \beta_0)$ for such β_0. As β is a limit of $\beta, (n-1)$-stables and therefore of $\beta, (n-2)$-stables, we can apply induction to infer that $\mathbb{H}(\beta_0)$ is Σ_{n-1}-elementary in $\mathbb{H}(\beta)$ relative to F for such β_0 and therefore γ holds in $(\mathbb{H}(\beta), F)$. So we have shown that in all cases that γ holds in $(\mathbb{H}(\beta), F)$, completing the proof that F is neutral for the Stability Predicate.

To verify replacement relative to F, we need only observe that the above implies that for each n, if α is Ord, n-stable (i.e., $\mathbb{H}(\alpha)$ is Σ_n elementary in $(V, \vec{H})$ where $\vec{H} = \{(\beta, H(\beta)) \mid \beta \text{ strong limit}\}$) then it remains so relative to F. $\square$

We now turn to extendibility for P.

Lemma 12 *Suppose that $\alpha < \beta$ belong to C and p belongs to $P(\alpha)$. Then p has an extension q in $P(\beta)$.*

Proof. By induction on β. The statement is immediate by induction if β is not a limit point of C.

Suppose that β is a limit point of C but $\mathbb{H}(\beta)$ is not 1-admissible. Then there is a closed unbounded subset D of $C \cap \beta$ of ordertype less than β whose intersection with each of its limit points $\gamma < \beta$ is Δ_1 definable over $\mathbb{H}(\gamma)$. We can assume that both α and the ordertype of D are less than the minimum of D. Now enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits. Note that for limit i, q_i is indeed a condition because $\mathbb{H}(\beta_i)$ is not 1-admissible. The union of the q_i's is the desired extension of p in $P(\beta)$.

Next suppose that $\mathbb{H}(\beta)$ is n-admissible but not $(n+1)$-admissible for some finite $n > 0$:

If β is a limit of β, n-stables then proceed as in the previous case: Choose a closed unbounded subset D of $C \cap \beta$ of ordertype less than β, consisting of β, n-stables, whose intersection with each of its limit points $\gamma < \beta$ is Δ_{n+1} definable over $\mathbb{H}(\gamma)$. Assume that both α and the ordertype of D are less than the minimum of D, enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits. For limit i, q_i is indeed a condition because $\mathbb{H}(\beta_i)$ is not $(n+1)$-admissible and, as β_i is a limit of β_i, n-stables and therefore $\mathbb{H}(\beta_i)$ is n-admissible, q_i is n-generic for $P(<\beta_i)$. The union of the q_i's is the desired extension of p in $P(\beta)$, as its n-genericity of this union follows from the n-genericity of the individual q_i's.

If β is not a limit of β, n-stables then β must have cofinality ω (else by the n-admissibility of $\mathbb{H}(\beta)$, we could find cofinally many β, n-stables using the fact that β has uncountable cofinality). If $\varphi = \forall x \psi$ with ψ $\Sigma_{n-1}(\mathbb{H}(\beta), \dot{f})$ is a $\Pi_n(\mathbb{H}(\beta), \dot{f})$ sentence and q is a condition in $P(<\beta)$ we say that q *decides* φ if q either forces φ or forces $\neg\psi(\sigma)$ for some $P(<\beta)$-name σ. It suffices to show that any condition q in $P(<\beta)$ can be extended to decide each of fewer than β-many Π_n sentences with parameters from $H(\beta)$, as given this, we can extend p in ω steps to a condition in $P(\beta)$ which is n-generic. To show this, first note that the n-admissibility of $\mathbb{H}(\beta)$ implies that there are cofinally-many $\delta < \beta$ which are limits of $\beta, (n-1)$-stables. Now let $(\varphi_i \mid i < \delta)$, $\delta < \beta$ enumerate the given collection of fewer than β-many Π_n sentences and let D consist of all $\gamma < \beta$ which are limits of $\beta, (n-1)$-stables and large enough so that $H(\gamma)$ contains both q and this enumeration. Extend q successively to elements q_i of $P(\gamma_i)$, where $\gamma_{i+1} \geq \gamma_i$ is the least element of D so that either q_i forces φ_i in $P(<\beta)$ or q_{i+1} forces $\neg\psi_i(\sigma_i)$ in $P(<\gamma_{i+1})$ (where $\varphi = \forall x \psi_i$) for some $P(<\gamma_{i+1})$-name σ_i, taking unions at limits. For limit i, $\mathbb{H}(\gamma_i)$ is not n-admissible as the set of $j < i$ such that q_j forces φ_j in $P(<\beta)$ can be treated as a parameter in $H(\gamma_i)$. And for limit i, γ_i is a limit of $\gamma_i, (n-1)$-stables. It follows that q_i is a condition for limit i. As $\mathbb{H}(\beta)$ is n-admissible, this construction results in a sequence of q_i's of length δ, whose union it the desired extension of q deciding all of the given Π_n sentences.

Finally, suppose that $\mathbb{H}(\beta)$ is n-admissible for every finite n. Choose D to be closed unbounded in β so that any $\gamma < \beta$ which is a limit point of D is a limit of γ, n-stables for every n. (Note that we may choose D to be any cofinal ω-sequence if β has cofinality ω.) Assume that α is less than the least element of D and enumerate D as $\beta_0 < \beta_1 < \cdots$. Then successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits, and note that for limit i, q_i is a condition because its n-genericity follows from the fact that β_i is a limit of β_i, n-stables. This also applies to the union of the q_i's, the desired extension q. $\square$

Lemma 13 *(In Gödel-Bernays class theory) Suppose that $T = \{(n, \alpha) \mid \alpha$ is Ord, n-stable$\}$ exists (equivalently, there is a satisfaction predicate for V). Then there is a P-generic which is definable over $(V, T, <)$ where $<$ is a wellorder of V of length Ord.*

Proof. As in the last case of the proof of the previous lemma, let D be closed unbounded in Ord so that if α is a limit point of D then α is a limit of α, n-stables for every n. Such a class D is definable over (V, T). Then take $F : \mathrm{Ord} \to 2$ to be the union of a sequence of conditions $p_0 \subseteq p_1 \subseteq \cdots$ where p_{i+1} is the $<$-least extension of p_i in $P(\alpha_{i+1})$ and the α_i's form the increasing enumeration of D. Then F is n-generic for $P(<\mathrm{Ord}) = P$ for each n. $\square$

Corollary 14 *(In Morse-Kelley class theory) There exists a function F : Ord $\to 2$ such that $V = L[F]$ and F is neutral for the Stability Predicate.*

V is generic over the Stable Core

Now fix a function $F : \mathrm{Ord} \to 2$ as in the last section, i.e., with the following properties:

1. $V = L[F]$, (V, F) satisfies replacement with a predicate for F.
2. If $n > 0$, α is β, n-stable and the $\beta, (n-1)$-stables are cofinal in β, then α is β, n-stable relative to F.

Now let M be a definable inner model of V. We describe a forcing Q definable over $(M[S], S)$ such that for some Q-generic G, G is definable over $(M[F], F)$ and F is definable over $(M[G], G)$. It follows that $M[G] = M[F]$, and as $L[F] = V$ we have $M[G] = V$.

The language $\mathcal{L}$ is defined inductively as follows, where $\dot{f}$ is a unary function symbol.

1. For each ordinal α, "$\dot{f}(\alpha) = 0$" and "$\dot{f}(\alpha) = 1$" are sentences of $\mathcal{L}$.
2. If Φ is a set of sentences of $\mathcal{L}$ and Φ belongs to $M[S]$, then $\bigwedge \Phi$ and $\bigvee \Phi$ are sentences of $\mathcal{L}$.

A sentence φ of $\mathcal{L}$ is *valid* if it is true when the symbol $\dot{f}$ is replaced by any function that belongs to a set-generic extension of $M[S]$. This notion is $M[S]$-definable and moreover if φ is a sentence of $\mathcal{L}$ and N is any outer model of $M[S]$, then φ is valid in $M[S]$ iff it is valid in N[22].

Let the definable inner model M be Σ_k-definable with parameter x. Now let T consist of all sentences of $\mathcal{L}$ of the form

$$\bigwedge(\Phi \cap H(\alpha)) \to \bigwedge(\Phi \cap H(\beta)),$$

where $\alpha < \beta$ are strong limit cardinals and for some $n > 0$:

(a) Φ is Σ_n definable over $H(\beta) \cap M[S]$ using parameters from $H(\alpha) \cap M[S]$ and the paramaeter x belongs to $H(\alpha)$.
(b) α and β are $\mathrm{Ord}, k+1$-stable (i.e., $\mathbb{H}(\alpha)$ and $\mathbb{H}(\beta)$ are Σ_{k+1}-elementary in $(V, H \restriction \mathrm{Ord})$ and α is nicely $\beta, (n+k+1)$-stable.

Note that (a),(b) imply that Φ is Σ_{n+k+1}-definable over $\mathbb{H}(\beta)$ (using parameters from the $H(\alpha)$ of V). It follows that the sentences in T are true when

[22] Indeed, if there is a function witnessing the non-validity of φ in a set-generic extension of N then we may assume that this generic extension is $N[G]$ where G is generic for a Lévy collapse making φ countable; then $M[S][G]$ also has a witness to the non-validity of φ, by Lévy absoluteness. Conversely, if the non-validity of φ is witnessed in a set-generic extension of $M[S]$ then this will happen in $M[S][G]$ where G is Lévy collapse generic over $M[S]$. Choose a condition in the Lévy collapse which forces this and choose H containing this condition which is Lévy collapse generic over N; then the non-validity of φ is witnessed in $N[H]$, a set-generic extension of N.

$\dot{f}$ is interpreted as F. Also note that T is $(M[S], S)$ definable, as (b) is expressed by the Stability Predicate S, using the following:

Fact. For each k, the class of strong limit cardinals which are Ord, k-stable is definable over $(L[S], S)$.

Proof. By induction on k. The base case $k = 0$ follows from the fact that every strong limit cardinal is $\mathrm{Ord}, 0$-stable. For any k, α is $\mathrm{Ord}, k+1$-stable iff for unboundedly many Ord, k-stable β greater than α, α is nicely β, k-stable. So by induction $\mathrm{Ord}, k+1$-stability is definable over $(L[S], S)$. $\square$ (*Fact*)

The desired forcing Q consists of all sentences φ of $\mathcal{L}$ which are consistent with T, in the sense that for no subset T_0 of T is the sentence $\bigwedge T_0 \to \neg\varphi$ valid[23]. The sentences in Q are ordered by: $\varphi \leq \psi$ iff T implies $\varphi \to \psi$.

Lemma 15 *Q has the Ord-chain condition, i.e., any $(M[S], S)$-definable maximal antichain in Q is a set.*

Proof. Suppose that A is an $(M[S], S)$-definable maximal antichain and consider $\Phi = \{\neg\varphi \mid \varphi \in A\}$. Then Φ is also $(M[S], S)$-definable. Choose n so that Φ is Σ_n-definable over $(M[S], S)$ and choose α to be Ord, n-stable and large enough so that $H(\alpha) \cap M[S]$ contains the parameters in the Σ_n definition of Φ. Then T together with $\Phi \cap H(\alpha)$ implies $\Phi \cap H(\beta)$ for all β greater than α which are Ord, n-stable and since there are arbitrarily large such β, T together with $\Phi \cap H(\alpha)$ implies all of Φ. It follows that A equals $A \cap H(\alpha)$: Otherwise let φ belong to $A \setminus H(\alpha)$. As $\neg\varphi$ belongs to Φ it is implied by T together with $\Phi \cap H(\alpha)$. But as A is an antichain, T together with φ implies $\Phi \cap H(\alpha)$ and therefore T together with φ implies $\neg\varphi$, contradicting the fact that φ belongs to Q. $\square$

[23] This idea of defining a forcing by taking the infinitary quantifier-free sentences consistent with a particular theory has appeared several times in the literature. In Proposition 5.31 of [5] it is used to study the possible genericity of $0^{\#}$ over an inner model not containing it, in [10] such a forcing is used to give a simple proof of Bukovsky's Theorem characterising set-genericity in terms of a strong covering property and Woodin's "Extender Algebra" (mentioned in the Introduction) provides a forcing of this form. A link between the latter two forcings is explored in [18].

Now it is easy to see that $V = M[F] = M[G]$ for some G which is Q-generic over $(M[S], S)$: Let G consist of all sentences in Q which are true when $\dot{f}$ is interpreted as F. It is obvious that G intersects all maximal antichains of Q which are sets in $M[S]$, as if the set A is an antichain missed by G then $\bigwedge\{\neg\varphi \mid \varphi \in A\}$ is consistent with T and witnesses the failure of A to be maximal. By Lemma 15 this gives full genericity over $(M[S], S)$.

This completes the proof of Theorem 10.

3. Strongly capturing the Stable Core

In this section we show that stability can be expressed in terms of strength in a definable iterate of Mighty Mouse, and therefore the Stable Core is definable over such an iterate.

We say that α is $\beta, 0$-*strong* if $\alpha < \beta$ are each uncountable cardinals. Let $\text{Strong}_0(\beta)$ denote the set of α which are $\beta, 0$-strong. The notions *nicely* $\beta, 0$-*strong* and $\beta, 0$-*correct* coincide with $\beta, 0$-strong.

An extender E with critical point less than a cardinal δ is $\delta, 1$-*strong* if $H(\delta)$ is contained in its ultrapower. α is $\beta, 1$-*strong* if α is $\beta, 0$-strong and for each $\beta, 0$-correct (i.e. $\beta, 0$-strong) δ greater than α there is an extender with critical point α which is $\delta, 1$-strong. α is *nicely* $\beta, 1$-*strong* if in addition β is a limit of $\beta, 0$-strongs[24]. And α is $\beta, 1$-*correct* if α is $\beta, 0$-correct and the $\alpha, 1$-strongs are the $\beta, 1$-strongs less than α. It follows from Lemma 17(b) below that if α is $\beta, 1$-strong and there is a $\beta, 0$-correct greater than α then α is also $\beta, 1$-correct. Let $\text{Strong}_1(\beta)$ denote the set of $\beta, 1$-strongs.

For $n > 0$, an extender E with critical point less than β is $\beta, n{+}1$-*strong* if it is β, n-strong and $j_E(\text{Strong}_n(\beta)) \cap \beta = \text{Strong}_n(\beta)$. α is $\beta, n{+}1$-*strong* if it is β, n-strong and for each β, n-correct $\bar{\beta}$ greater than α, there is an extender with critical point α which is $\bar{\beta}, n+1$-strong. α is *nicely* $\beta, n+1$-*strong* if in addition the β, n-strongs are cofinal in β[25], and α is $\beta, n+1$-*correct* if it is β, n-correct and the $\alpha, n+1$-strongs are the $\beta, n+1$-strongs less than α.

[24] Note that if α is $\beta, 1$-strong then whether or not it is nicely $\beta, 1$-strong depends only on β and not on α.

[25] As in the case $n = 0$, if α is $\beta, n+1$-strong then whether or not it is nicely $\beta, n+1$-sttrong depends only on β and not on α.

It follows from Lemma 17(b) below that if α is $\beta, n+1$-strong and there is a β, n-correct greater than α then α is also $\beta, n+1$-correct. Let $\mathrm{Strong}_{n+1}(\beta)$ denote the set of $\beta, n+1$-strongs. We say that α is *β-correct* if it is β, n-correct for every n.

Our aim is to show that there is a definable iterate mm* of Mighty Mouse such that nice β, n-stability (in V) can be expressed in terms of measurability and strength properties in mm*. This is made possible by the fact that many of the basic properties of β, n-stability transfer to β, n-strength, as indicated in the next lemmas.

Lemma 16 *(Stability Lemma) (a) If α is $\beta, n+1$-stable then it is also nicely β, n-stable.*
(b) If α is $\beta, n+1$-stable and β is $\gamma, n+1$-stable then α is $\gamma, n+1$-stable[26]*.*
(c) If α is $\gamma, n+1$-stable and β is γ, n-stable with $\alpha < \beta$, then α is $\beta, n+1$-stable.
(d) If α is $\beta, n+1$-stable then α is a limit of α, n-stables.
(e) If γ is a limit of γ, n-stables and α is $\beta, n+1$-stable whenever β is γ, n-stable and greater than α then α is $\gamma, n+1$-stable.

Proof. As (b), (c) and (e) are rather obvious, we prove only (a) and (d).

(a) We may assume $n > 0$. If some α is $\beta, n+1$-stable we have to check that β is a limit of $\beta, n-1$-stables. As $n-1$-stability is a Π_{n-1} property, the boundedness of the $n-1$-stables is a Σ_{n+1} property. So if the $\beta, n-1$-stables are bounded in β, so are the $\alpha, n-1$-stables bounded in α. But then the largest $\alpha, n-1$-stable is also the largest $\beta, n-1$-stable, contradicting the fact that α is $\beta, n-1$-stable.

(d) n-stabiliy is a Π_n property. For any $\bar{\alpha} < \alpha$ there is a β, n-stable greater than $\bar{\alpha}$ (namely α). As α is $\beta, n+1$-stable, there is an α, n-stable greater than $\bar{\alpha}$. $\square$

Some analogous properties for β, n-strength are contained in the next lemma.

Lemma 17 *(Strength Lemma) The following properties hold for arbitrary uncountable cardinals:*

[26] This is also true with $n+1$ replaced by 0, but we write it with $n+1$ instead of n to maintain an analogy with part (b) of the Strength Lemma below.

(a) If α is $\beta, n+1$-strong then it is also β, n-strong.
(b) If α is $\beta, n+1$-strong, β is $\gamma, n+1$-strong and there is a γ, n-correct greater than β then α is $\gamma, n+1$-strong. If β is $\gamma, n+1$-strong and there is a γ, n-correct greater than β then β is $\gamma, n+1$-correct.
(c) If α is $\gamma, n+1$-strong, β is γ, n-strong, $\alpha < \beta$ and (if $n > 0$) there is a $\gamma, n-1$-correct greater than β then α is $\beta, n+1$-strong.
(d) If α is $\beta, n+1$-strong and there is a β, n-correct greater than α then α is a limit of α, n-strongs.
(e) If γ is a limit of γ, n-corrects and α is $\beta, n+1$-strong whenever β is γ, n-correct and greater than α then α is $\gamma, n+1$-strong.

Proof. By induction on n. For the base case $n = 0$:

(a) This is immediate by the definition of $\beta, 1$-strength.

(b) Suppose that α is less than $\bar{\gamma}$ and $\bar{\gamma}$ is a cardinal less than γ; we want to show that there is an extender with critical point α which is $\bar{\gamma}, 1$-strong, i.e., which has $H(\bar{\gamma})$ in its ultrapower. If $\bar{\gamma}$ is less than β then since α is $\beta, 1$-strong we get the desired extender.

If $\bar{\gamma}$ equals β then by hypothesis we can choose a cardinal δ between β and γ; as β is $\gamma, 1$-strong we can choose an extender E with critical point β which is $\delta, 1$-strong. By elementarity, α is $j_E(\beta), 1$-strong in M_E. In M_E pick any cardinal β^* between β and $j_E(\beta)$. As α is $j_E(\beta), 1$-strong in M_E, there is an extender with critical point α which is $\beta^*, 1$-strong in M_E. This extender is also $\beta, 1$-strong.

Finally, suppose that $\bar{\gamma}$ is greater than β. As β is $\gamma, 1$-strong there is an extender E with critical point β which is $\bar{\gamma}, 1$-strong. α is $j_E(\beta), 1$-strong in M_E. Choose any M_E-cardinal β^* between $\bar{\gamma}$ and $j_E(\beta)$. Then there is an extender with critical point α which is $\beta^*, 1$-strong. This extender is also $\bar{\gamma}, 1$-strong.

For the second statement of (b) note that if α is $\beta, 1$-strong then by the first statement of (b) it is $\gamma, 1$-strong; conversely any $\gamma, 1$-strong less than β is by definition $\beta, 1$-strong as β is $\gamma, 0$-correct.

(c) α is $\beta, 1$-strong because it is $\gamma, 1$-strong and every $\beta, 0$-correct is also $\gamma, 0$-correct.

(d) The hypothesis implies that there is $j : V \to M$ with critical point α. This implies that α is a limit cardinal and therefore a limit of $\alpha, 0$-strongs.

(e) Suppose that $\bar{\gamma}$ is a cardinal less than γ and greater than α. Choose β greater than $\bar{\gamma}$ which is $\gamma, 0$-correct. As α is $\beta, 1$-strong there is an extender with critical point α which is $\bar{\gamma}, 1$-strong.

Now suppose that $n > 0$ and (a) - (e) hold for $n - 1$; we verify (a) - (e) for n.

(a) Again this follows from the definition of $\beta, n+1$-strength.

(b) For the first statement, we know that α is γ, n-strong by induction (and the fact that γ, n-correctness implies $\gamma, n-1$-correctness, by definition). Suppose that $\alpha < \bar{\gamma}$ and $\bar{\gamma}$ is γ, n-correct. If $\bar{\gamma}$ is less than β then by (b) for $n-1$, β is γ, n-correct and therefore $\bar{\gamma}$ is β, n-correct. As α is $\beta, n+1$-strong we get the desired $\bar{\gamma}, n+1$-strong extender with critical point α.

If $\bar{\gamma}$ equals β then let δ be γ, n-correct and greater than β; as β is $\gamma, n+1$-strong we can choose an extender E with critical point β which is $\delta, n+1$-strong. Then α is $j_E(\beta), n+1$-strong in M_E and the β, n-strongs are the $j_E(\beta), n$-strongs in M_E less than β. It follows that β is $j_E(\beta), n$-correct in M_E and therefore there is an extender with critical point α which is $\beta, n+1$-strong.

Finally, suppose that $\bar{\gamma}$ is greater than β. As β is $\gamma, n+1$-strong it is also γ, n-strong and therefore by (b) for $n-1$ it is γ, n-correct and therefore $\bar{\gamma}, n$-correct. As β is $\gamma, n+1$-strong there is an extender E with critical point β which is $\bar{\gamma}, n+1$-strong. Thus in M_E, $\bar{\gamma}$ is $j_E(\bar{\gamma}), n$-correct. Also by elementarity, $j_E(\beta)$ is $j_E(\bar{\gamma}), n$-correct so we have that $\bar{\gamma}$ is $j_E(\beta), n$-correct in M_E. Also by elementarity, α is $j_E(\beta), n+1$-strong in M_E and therefore there is an extender with critical point α which is $\bar{\gamma}, n+1$-strong.

For the second statement of (b) note that if α is $\beta, n+1$-strong then by the first statement of (b) it is $\gamma, n+1$-strong; conversely any $\gamma, n+1$-strong less than β is by definition $\beta, n+1$-strong as by (b) for the case $n-1$, β is γ, n-correct.

(c) By (b) for the case $n-1$, β is γ, n-correct. It then follows from the $\gamma, n+1$-strength of α that α is $\beta, n+1$-strong.

(d) Choose a β, n-correct $\bar{\beta}$ which is greater than α. By (b) for $n-1$, α is β, n-correct and therefore $\bar{\beta}, n$-correct. As α is $\beta, n+1$-strong there is an extender E with critical point α which is $\bar{\beta}, n+1$-strong. So α is $j_E(\bar{\beta}), n$-strong in

M_E. If α were not a limit of α, n-strongs then α could not be $j_E(\alpha), n$-strong in M_E, contradicting the fact that it is $j_E(\bar{\beta}), n$-strong in M_E and $j_E(\alpha)$ is $j_E(\bar{\beta}), n$-correct in M_E.

(e) Let $\bar{\gamma}$ be γ, n-correct and greater than α. Choose β greater than $\bar{\gamma}$ which is γ, n-correct. Then $\bar{\gamma}$ is also β, n-correct. As α is $\beta, n+1$-strong, there is an extender with critical point α which is $\bar{\gamma}, n+1$-strong. $\square$

Remarks. (a) In Lemma22(a) below, we show that any limit of strong limits β is a limit of β-corrects. It follows that the hypotheses concerning the existence of γ-corrects in Lemma17(b),(c) can be dropped if γ is a limit of strong limits and the hypothesis concerning the existence of *beta*-corrects in Lemma17(d) can be dropped if β is a limit of strong limits. (b) we point out a dissimilarity between stability and strength: If α is $\beta, n+2$-stable then the β, n-stables are cofinal in β. The analogous statement fails for strength. However there is a variant of stability called pseudostability which we introduce later in this section and which serves as a better analogue of strength.

In what follows we will apply the above discussion of β, n-strength not to this notion in V, but to this notion as interpreted by a mouse (or weasel, i.e., a mouse of height Ord). Lemma 17 is valid for this "localised" notion as well.

The iteration

We define an iteration $((m_\gamma, E_\gamma) \mid \gamma \in \mathrm{Ord})$ of mm = Mighty Mouse where E_γ is a total extender of m_γ. We set $m_0 = \mathrm{mm}$ and obtain $m_{\gamma+1} = \mathrm{Ult}(m_{\bar{\gamma}}, E_\gamma)$ by applying E_γ to $m_{\bar{\gamma}}$ where $\bar{\gamma}$ is least so that this ultrapower makes sense (i.e., so that $m_{\bar{\gamma}}$ and m_γ have the same subsets of the critical point of E_γ), thereby forming an iteration tree $\mathcal{T}$. At limit stages γ, m_γ is the direct limit of the $m_{\bar{\gamma}}$'s for $\bar{\gamma}$ in the unique cofinal branch through the iteration tree below γ (we will prove uniqueness) which by iterability yields a wellfounded direct limit m_γ. Therefore the entire iteration is uniquely determined by the choice of E_γ's.

But before specifying the E_γ's we first have to refine the notion of β, n-stability.

Recall that for strong limits $\alpha < \beta$, α is $\beta, 1$-stable if $\mathbb{H}(\alpha)$ is Σ_1-elementary in $\mathbb{H}(\beta)$. A *beth-number* is a cardinal of the form $\beth_\alpha$ for some ordinal α where $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\lambda = \cup_{\alpha<\lambda}\beth_\alpha$ for limit λ. The (uncountable) strong limits are the limits of beth-numbers. Now for an arbitrary beth-number α we define $\mathbb{H}(\alpha)$ to be $(H(\alpha), H||\alpha)$ where $H||\alpha$ denotes the set of pairs $(\bar{\alpha}, H(\bar{\alpha}))$ such that $H(\bar{\alpha})$ belongs to $H(\alpha)$. Then $\mathbb{H}(\alpha)$ is Σ_0-elementary in $\mathbb{H}(\beta)$ for beth-numbers $\alpha < \beta$. For beth-numbers $\alpha < \beta$, we say that α is $\beta, 1$-*stable* if $\mathbb{H}(\alpha)$ is Σ_1-elementary in $\mathbb{H}(\beta)$. This implies that α is strong limit, but can hold if β is a successor beth-number.

Regarding $\beta, n+1$-stability for $n > 0$ we introduce *pseudostability*. Recall that if the $\beta, 1$-stables are cofinal in β then α is $\beta, 2$-stable iff $(\mathbb{H}(\alpha), \mathrm{Stable}_{\leq 1}(\alpha))$ is Σ_1-elementary in $(\mathbb{H}(\beta), \mathrm{Stable}_{\leq 1}(\beta))$, where $\mathrm{Stable}_{\leq 1}(\alpha)$ refers to the pair of predicates $\mathrm{Stable}_0(\alpha), \mathrm{Stable}_1(\alpha)$. If we drop the assumption that the $\beta, 1$-stables are cofinal in β then we refer to this notion as $\beta, 2$-*pseudostability*. More generally, if $\alpha < \beta$ are beth numbers we say that α is $\beta, n+1$-*pseudostable* if $(\mathbb{H}(\alpha), \mathrm{PStable}_{\leq n}(\alpha))$ is Σ_1-elementary in $(\mathbb{H}(\beta), \mathrm{PStable}_{\leq n}(\beta))$, where $\mathrm{PStable}_{\leq n}(\alpha)$ refers to the sequence of predicates $\mathrm{PStable}_0(\alpha), \mathrm{PStable}_1(\alpha)$, $\cdots, \mathrm{PStable}_n(\alpha)$ and $\mathrm{PStable}_i(\alpha)$ is the set of α, i-pseudostables. (For $i = 0, 1$, $\mathrm{PStable}_i(\alpha) = \mathrm{Stable}_i(\alpha)$.)

The reasons for extending the notion of 1-stability to beth-numbers and for introducing pseudostability is that to inductively calibrate strength with stability at strong limits we need to approximate strong limits by arbitrary beth-numbers and apply a notion of n-stability that applies to them which more closely resembles the notion of strength (see Lemma 22(b),(c) below).

We also consider *pseudostable-correctness*. For beth-numbers $\alpha < \beta$ and finite $n > 0$, we say that α is β, n-*ps-correct* (β, n-*pseudostable-correct*) if the α, n-pseudostables are the β, n-pseudostables less than α. And α is β-*ps-correct* (β-*pseudostable-correct*) if α is β, n-ps-correct for each n.

As the E_γ's used in our iteration will be chosen to be total extenders, each resulting iterate m_γ has a largest cardinal which carries a normal measure in m_γ (as a predicate). We denote this by κ_γ.

Now we are prepared to define the extenders E_γ to be used in our iteration. E_γ will either be a total measure in m_γ or a total extender in m_γ which is not a measure and has length a beth-number (of V); we use β_γ to denote the critical point of E_γ if it is a measure and otherwise the length of E_γ. We say that $\beta < \kappa_\gamma$ is *worrisome for* m_γ if one of the following holds.

1. β is a successor beth-number in V and for some n, some α is not $\beta, n+1$-pseudostable in V yet there is a $\beta, n+1$-strong extender in m_γ with critical point α.

2. For some n, β is measurable, $\kappa_\gamma, n+1$-correct and the limit of β, n-strongs in m_γ but not a limit of β, n-stables in V[27].

For the least worrisome $\beta = \beta_\gamma$ in m_γ, if Case 1 holds then we choose the least n that witnesses Case 1 and the least α that witnesses Case 1 for n and let E_γ be the $\beta_\gamma, n+1$-strong extender in m_γ with critical point α of least index. If Case 1 fails but Case 2 holds then E_γ is the order 0 measure on β_γ in m_γ.

of pseudostability by failures of strength. Case 2 is needed to show that for strong limit β, the beth-number predecessors of β on the iteration tree are those which are β-ps-correct (see Lemma 22 (d)). For this purpose we need to know that a strong limit β is sufficiently κ_β-correct in m_β. To illustrate, suppose that $\bar\beta < \beta$ are adjacent strong limits and $\bar\beta$ is not $\bar\beta^*, 1$-ps-correct where $\bar\beta^*$ is the least beth-number greater than $\bar\beta$; we need to ensure that $\bar\beta$ is not below β on the iteration tree. Let α be the least $\bar\beta, 1$-stable that is not $\bar\beta^*, 1$-stable. By Lemma 22(c) α is $\bar\beta, 1$-strong in $m_{\bar\beta}$. Thanks to the $\kappa_{\bar\beta}, 1$-correctness of $\bar\beta$ in $m_{\bar\beta}$, it follows that α is also $\kappa_{\bar\beta^*}, 1$-strong in $m_{\bar\beta^*}$ and therefore there is an extender with critical point α and length $\bar\beta^*$ in $m_{\bar\beta^*}$. Such an extender can be applied via Case 1 of "worrisome" and as this extender (as well as any extender used in the iteration between β^* and β which is not a measure) "overlaps" $\bar\beta$, the latter is not below β in the iteration tree.

If m_γ is worry-free (i.e., no $\beta < \kappa_\gamma$ is worrisome for m_γ) then we set $\beta_\gamma = \kappa_\gamma$, the largest measurable of m_γ, and take E_γ to be the normal measure on β_γ in m_γ (as a predicate).

[27] A limit of $\beta, 0$-strongs is the same as a limit of uncountable cardinals, i.e. an uncountable limit cardinal. A limit of $\beta, 0$-stables is the same as a limit of beth-numbers, i.e. a strong limit.

This completes the definition of the iteration. For any γ, let i_γ denote the index of the extender E_γ in m_γ. Also let $\mathcal{T}$ denote the iteration tree that results from this iteration. (Recall that the immediate $\mathcal{T}$-predecessor of $\gamma+1$ is the least δ such that E_γ can be applied to m_δ, and then $m_{\gamma+1} = \mathrm{Ult}(m_\delta, E_\gamma)$.)

We make the following useful observation about the iteration tree.

Proposition 18 *If E_γ is an extender of length γ (i.e. not a measure) then crit(E_γ) is the immediate predecessor of $\gamma+1$ in the iteration tree $\mathcal{T}$.*

Proof. All subsets of $\mathrm{crit}(E_\gamma)$ in $m_{\mathrm{crit}(E_\gamma)}$ belong to m_γ so E_γ can be applied to $m_{\mathrm{crit}(E_\gamma)}$. And if α is less than $\mathrm{crit}(E_\gamma)$ then E_α is coded by a (bounded) subset of $\mathrm{crit}(E_\gamma)$ which does not belong to m_α and therefore E_γ cannot be applied to m_α. It follows that $\mathrm{crit}(E_\gamma)$ is the $\mathcal{T}$-predecessor of $\gamma+1$. $\square$

Lemma 19 *The β_γ's are strictly increasing. Also, the indices i_γ are strictly increasing.*

Proof. Suppose that γ_0 is the immediate tree-predecessor of γ_1+1. If E_{γ_1} is a measure then $\gamma_0 = \gamma_1$ and the embedding from m_{γ_1} to m_{γ_1+1} sends β_{γ_1} to an ordinal greater than β_{γ_1}. If E_{γ_1} is not a measure then the embedding from m_{γ_0} to m_{γ_1+1} sends β_{γ_0} to an ordinal greater than β_{γ_1}. As in both cases the embedding from m_{γ_0} to m_{γ_1+1} sends β_γ to β_{γ_1+1} it follows that β_{γ_1+1} is greater than β_{γ_1}.

For γ limit, if β_γ were less than the supremeum of the $\beta_{\bar\gamma}$, $\bar\gamma < \gamma$, then it would be less than the critical point of the embedding from $m_{\bar\gamma}$ to m_γ for some $\bar\gamma$ below γ on the iteration tree. But by elementarity, β_γ is worrisome for $m_{\bar\gamma}$, contradicting the assumption that it is less than $\beta_{\bar\gamma}$.

As the index i_γ of E_γ lies between β_γ and $\beta_{\gamma+1}$ it follows that the i'_γ are strictly increasing as well. $\square$

Corollary 20 *m_0^* is worry-free.*

Next we establish the desired connection between $\beta, n+1$-pseudostability in V and $\beta, n+1$-strength in m_0^*. First we identify the strong limits (of V) in m_0^*.

Lemma 21 *The measurable* $Ord, 1$*-corrects of* m_0^**, together with their limits, are strong limits in* V*. Conversely, each strong limit of* V *is a measurable* $Ord, 1$*-correct of* m_0^* *or the limit of measurable* $Ord, 1$*-corrects of* m_0^**.*

Proof. The first statement follows from the fact that m_0^* is worry-free. For the second statement, if α is not the limit of strong limits let $\bar{\alpha}$ be the largest strong limit less than α ($\bar{\alpha} = 0$ if α is the least strong limit). Then $E_{\bar{\alpha}+1}$ is the measure on the least $\kappa_{\bar{\alpha}+1}, 1$-correct measurable of $m_{\bar{\alpha}+1}$ greater than $\bar{\alpha}$. At any stage β between $\bar{\alpha}$ and α which is not a beth-number the measure in m_β on the least $\kappa_\beta, 1$-correct measurable greater than $\bar{\alpha}$ is applied. Thus α is measurable and $\kappa_\alpha, 1$-correct in m_α. α remains measurable and Ord, 1-correct in m_0^*. □

The desired connection between $\beta, n+1$-pseudostability and $\beta, n+1$-strength is contained in the following lemma (where "correct, strong" refer to these notions in m_0^* and "pseudostabie, ps-correct" refer to these notions in V).

Lemma 22 *(Main Lemma) For* β *strong limit and* n *finite:*
(a) If β *is not a limit of strong limits then all suffciently large beth-numbers less than* β *are both* β*-correct and* β*-ps-correct. If* β *is a limit of strong limits then cofinally-many strong limits less than* β *are both* β*-correct and* β*-ps-correct.*
(b) A $\beta, n+1$*-strong is also* $\beta, n+1$*-pseudostable.*
(c) A successor $\beta, n+1$*-pseudostable is also* $\beta, n+1$*-strong.*
(d) There is a unique branch b_β *cofinal in the iteration tree below* β*. A beth-number belongs to* b_β *iff it is* β*-ps-correct.*
(e) β *is* $\kappa_\beta, 1$*-correct in* m_β*. If* β *is a limit of* β, n*-stables then* β *is* $\kappa_\beta, n+1$*-correct in* m_β*.*

Proof. By induction on β and for fixed β by induction on n. In proving (a) for β we use (b),(c) for smaller $\bar{\beta}$, in proving (b) we use (a), in proving (c) for β we use (d) for smaller $\bar{\beta}$, in proving (d) for β we use (b), (c), (d) and (e) for smaller $\bar{\beta}$ and in proving (e) we use (d).

(a) For the first statement, we first prove by induction on n that all sufficiently large beth-numbers less than β are $\beta, n+1$-correct. For $n = 0$, if γ_0 is greater than all strong limits less than β and not $\beta, 1$-correct let α_0 be the least $\gamma_0, 1$-strong that is not $\beta, 1$-strong. Choose $\gamma_1 > gamma_0$ so that α_0 is not $\gamma_1, 1$-strong. If γ_1 is not $\beta, 1$-correct then let α_1 be the least $\gamma_1, 1$-strong that is not $\beta, 1$-strong. Then α_1 is not greater than α_0 as it is less than γ_0 (since α_1 is strong limit) and therefore by transitivity, (Lemma 17(b)) α_0 would be $\gamma_1, 1$-strong, contradiction. And α_1 cannot equal α_0 as α_0 is not $\gamma_1, 1$-strong but α_1 is $\gamma_1, 1$-strong. So α_1 is less than α_0. If γ_1 is not $\beta, 1$-correct then similarly choose γ_2 and α_2, arguing that α_2 is less than α_1. After finitely many steps one reaches a γ_k that is $\beta, 1$-correct, and all beth-numbers between γ_k and β are $\beta, 1$-correct as well.

If γ_k is not $\beta, 2$-correct then choose the least α_k which is $\gamma_k, 2$-strong but not $\beta, 2$-strong. Then choose γ_{k+1} greater than γ_k so that α_k is not $\gamma_{k+1}, 2$-strong. If γ_{k+1} is not $\beta, 2$-correct then let α_{k+2} be the least $\gamma_{k+1}, 2$-strong that is not $\beta, 2$-strong; α_{k+2} is less than α_{k+1} else α_{k+1} would be $\gamma_{k+2}, 2$-strong (since γ_{k+1} is $\gamma_{k+2}, 1$-correct and greater than $\alpha_{k+1)}$, a contradiction). Continue choosing $\gamma_{k+3}, \gamma_{k+4}, \dots$ until reaching a γ_l that is $\beta, 2$-correct. All beth-numbers between γ_l and β are also $\beta, 2$-correct. Continue in this way and we see that for each n, all sufficiently large beth-numbers less than β are $\beta, n+1$-correct.

Now argue as follows. Choose a beth-number γ_0 greater than all strong limits less than β. If γ_0 is not β-correct then let n_0 be least so that γ_0 is not β, n_0+1-correct and let α_0 be the least γ_0, n_0+1-strong that is not β, n_0+1-strong. Choose a beth-number $\gamma_1 > \gamma_0$ which is β, n_0+1-correct so that α_0 is not γ_1, n_0+1-strong. If γ_1 is not β-correct then choose the least n_1 so that γ_1 is not β, n_1+1-correct and the least α_1 which is γ_1, n_1+1-strong but not β, n_1+1-strong. Note that n_1 is greater than n_0 and therefore α_1 cannot be greater than α_0 else the latter would be γ_1, n_0+1-strong by transitivity, contradiction. And α_1 is not equal to α_0 so α_1 is less t4han α_0. If γ_1 is not β-correct then similarly choose γ_2 and α_2 (if γ_2 is not β-correct); as the α's decrease one reaches a γ which is β-correct. For each n, the γ, n-strongs equal the β, n-strongs. Then an induction shows that all beth-numbers between γ and β are β-correct.

For ps-correctness use the same argument, but now transitivity is automatic, so the argument is easier.

For the second statement of (a), we show by induction on n that the $\beta, n+1$-doubly-correct strong limits are cofinal in β (where *doubly-correct* means both correct and ps-correct). For $n = 0$ suppose that γ is a strong limit less than β. If γ is not $\beta, 1$-doubly-correct then let α be least so that α is either $\gamma, 1$-strong but not $\beta, 1$-strong or $\gamma, 1$-stable but not $\beta, 1$-stable.

If α is $\gamma, 1$-strong but not $\beta, 1$-strong let $\bar{\beta}$ be the least strong limit so that α is not $\bar{\beta}, 1$-strong. Then $\bar{\beta}$ is $\beta, 1$-correct: If $\bar{\alpha}$ is $\bar{\beta}, 1$-strong then $\bar{\alpha}$ cannot be greater than α (as $\bar{\beta}$ is a limit of $\bar{\beta}, 0$-corrects) and cannot equal α, so it must be less than α; but then by the minimality of α, $\bar{\alpha}$ must be $\beta, 1$-strong. But also $\bar{\beta}$ is $\beta, 1$-ps-correct: Suppose $\bar{\alpha}$ were the least $\bar{\beta}, 1$-stable which is not $\beta, 1$-stable. Then $\bar{\alpha}$ is a successor $\bar{\beta}, 1$-stable and hence by (c) for $\bar{\beta}$ is $\bar{\beta}, 1$-strong. Now $\bar{\alpha}$ cannot be less than α, else by the minimality of α it would be $\beta, 1$-stable, it cannot equal $\bar{\alpha}$ as it is $\bar{\beta}, 1$-strong and it cannot be greater than α else α would be $\bar{\beta}, 1$-strong.

Now suppose α is $\gamma, 1$-stable but not $\beta, 1$-stable. Let $\bar{\beta}$ be the least strong limit so that α is not $\bar{\beta}, 1$-stable. Then $\bar{\beta}$ is $\beta, 1$-ps-correct (the argument is similar to the argument above that $\bar{\beta}$ is $\beta, 1$-correct). But also $\bar{\beta}$ is $\beta, 1$-correct: Suppose $\bar{\alpha}$ is $\bar{\beta}, 1$-strong. Then $\bar{\alpha}$ is $\bar{\beta}, 1$-stable and therefore less than α. But by the minimality of α, γ is then $\beta, 1$-strong.

For $n > 0$ we proceed inductively. Suppose that γ is a strong limit less than β. If γ is not $\beta, n+1$-doubly-correct then let α be least so that α is either $\gamma, n+1$-strong but not $\beta, n+1$-strong or $\gamma, n+1$-pseudostable but not $\beta, n+1$-pseudostable.

If α is $\gamma, n+1$-strong but not $\beta, n+1$-strong let $\bar{\beta}$ be the least β, n-correct strong limit such that α is not $\bar{\beta}, n+1$-strong. Then $\bar{\beta}$ is $\beta, n+1$-correct: If $\bar{\alpha}$ is $\bar{\beta}, n+1$-strong then $\bar{\alpha}$ cannot be greater than α (as $\bar{\beta}$ is a limit of $\bar{\beta}, n$-corrects) and cannot equal α, so it must be less than α; but then by the minimality of α, $\bar{\alpha}$ must be $\beta, n+1$-strong. But also $\bar{\beta}$ is $\beta, n+1$-ps-correct: Suppose $\bar{\alpha}$ were the least $\bar{\beta}, n+1$-pseudostable which is not $\beta, n+1$-pseudostable. Now $\bar{\alpha}$ cannot be less than α, else by the minimality of α it would be $\beta, n+1$-pseudostable. Note that $\bar{\alpha}$ is a successor $\bar{\beta}, n+1$-pseudostable and hence by (c) for $\bar{\beta}$ is $\bar{\beta}, n+1$-strong. It follows that it cannot equal $\bar{\alpha}$ as it is $\bar{\beta}, n+1$-strong. And $\bar{\alpha}$ cannot be greater than α, else α would be $\bar{\beta}, n+1$-strong.

Now suppose α is $\gamma, n+1$-pseudostable but not $\beta, n+1$-pseudostable. Let $\bar{\beta}$ be the least β, n-ps-correct strong limit so that α is not $\bar{\beta}, n+1$-pseudostable. Then $\bar{\beta}$ is $\beta, n+1$-ps-correct (the argument is similar to the argument above that $\bar{\beta}$ is $\beta, 1$-ps–correct). But also $\bar{\beta}$ is $\beta, n+1$-correct: Suppose $\bar{\alpha}$ is $\bar{\beta}, n+1$-strong. Then $\bar{\alpha}$ is $\bar{\beta}, n+1$-pseudostable and therefore less than α. But by the minimality of α, $\bar{\alpha}$ is then $\beta, n+1$-strong.

(b) If γ is not $\beta, n+1$-pseudostable then we can choose a successor beth-number $\bar{\beta}$ less than β which is both β-correct and β-ps-correct such that γ is not $\bar{\beta}, n+1$-pseudostable but is $\bar{\beta}, n+1$-strong. We may assume that $E_{\bar{\beta}}$ is a measure on $\bar{\beta}$ and therefore $\bar{\beta}$ is weakly compact in m_0^*. So by reflection there is a $\beta, n+1$-strong extender with critcal point γ. This contradicts the fact that m_0^* is worry-free.

(c) By (d) for $\bar{\beta}$, if $\bar{\beta}$ is a successor $\beta, n+1$-pseudostable then the beth-numbers in $b_{\bar{\beta}}$ are the $\bar{\beta}$-ps-correct beth-numbers and therefore $b = b_{\bar{\beta}}$ is Σ_n definable over $\mathbb{H}(\bar{\beta})$. It follows that for cofinally-many α in b, E_α has critical point α and length greater than any beth-number δ such that α is $\delta, n+1$-pseudostable. In particular, such α are $\delta, n+1$-strong whenever α is $\delta, n+1$-pseudostable. It follows from the $\beta, n+1$-pseudostability of $\bar{\beta}$ that also $\bar{\beta}$ is $\delta, n+1$-strong whenever it is $\delta, n+1$-pseudostable, and setting $\delta = \beta$ we conclude that $\bar{\beta}$ is $\beta, n+1$-strong.

(d) Suppose that β is a limit of strong limits. Let b be any branch cofinal through the iteration tree below β. Then cofinally-many elements of b are strong limits: If cofinally-many of the extenders used along b are not measures then their critical points form a set of strong limits in b cofinal in β, and otherwise only measures are used on a final segment of b, in which case all sufficiently large strong limits belong to b. By induction if $\bar{\beta}_0 < \bar{\beta}_1$ are strong limits in b then $\bar{\beta}_0$ is $\bar{\beta}_1$-ps-correct; it follows that each strong limit in b is β-ps-correct and again by induction, the beth-numbers in b are the β-ps-corrects.

Suppose that β is a successor strong limit and $\bar{\beta}$ its strong limit predecessor ($= 0$ if β is the least strong limit). Suppose that $\bar{\beta}$ is β-ps-correct (or $\bar{\beta} = 0$). Then for no beth-number γ in the interval $(\bar{\beta}, \beta)$ does E_γ overlap $\bar{\beta}$ (i.e. have critical point less than $\bar{\beta}$): Otherwise the critical point κ of E_γ witnesses the failure of $\bar{\beta}$ to be β-ps-correct. It follows that $\bar{\beta}$ belongs to b_β and the beth-numbers in b_β are the $\bar{\beta}$-ps-corrects = the β-ps-corrects less than $\bar{\beta}$ together with the β-ps-corrects greater than $\bar{\beta}$. Conversely, if $\bar{\beta}$ is not β-ps-correct, choose the least n so that $\bar{\beta}$ is not $\beta, n+1$-ps-correct and $\kappa < \bar{\beta}$ least which is $\bar{\beta}, n+1$-pseudostable but not $\beta, n+1$-pseudostable. Note that κ is a successor $\bar{\beta}, n+1$-pseudostable and therefore by (c) for $\bar{\beta}$ is $\bar{\beta}, n+1$-strong. As $\bar{\beta}$ is a limit of $\bar{\beta}, n$-stables (using the minimality of

n and (b) for $\bar{\beta}$) it follows from (e) for $\bar{\beta}$ that $\bar{\beta}$ is $\kappa_{\bar{\beta}}, n+1$-correct in $m_{\bar{\beta}}$. So there is an extender E in $m_{\bar{\beta}}$ with critical point κ and length the least $\kappa_{\bar{\beta}}, n+1$-correct measurable in $m_{\bar{\beta}}$ greater than $\bar{\beta}$. If $\bar{\beta}^*$ is the least beth-number greater than $\bar{\beta}$ such that κ is not $\bar{\beta}^*, n+1$-pseudostable then the iteration from $m_{\bar{\beta}}$ to $m_{\bar{\beta}^*}$ sends E to an extender E^* of length at least $\bar{\beta}^*$ whose truncation $E^*|\bar{\beta}^*$ is $\bar{\beta}^*, n+1$-strong. As $\kappa =$ the critical point of E^* is not $\bar{\beta}^*, n+1$-pseudostable, an extender is applied at stage $\bar{\beta}^*$ of the iteration which overlaps $\bar{\beta}$. If $\bar{\beta}^*$ is the largest beth-number stage less than β where a non-measure is applied, then $\bar{\beta}^*+1$ belongs to b_β and witnesses that $\bar{\beta}$ does not belong to b_β. Otherwise $\bar{\beta}^{**}+1$ belongs to b_β for some beth-number $\bar{\beta}^{**}$ greater than $\bar{\beta}^*$ where the critical point of $E_{\bar{\beta}^{**}}$ is less than κ, again showing that $\bar{\beta}$ does not belong to b_β. And the largest β-ps-correct less than $\bar{\beta}$ is the critical point $\bar{\bar{\beta}}$ of E_γ where γ is the largest beth-number less than β where E_γ is not a measure. And the beth-numbers in b_β are the β-ps-corrects less than β which are either less than or equal to $\bar{\bar{\beta}}$ or are at least γ.

(e) For the first statement, if $\bar{\beta}$ is a limit of strong limits then we can choose a strong limit $\bar{\bar{\beta}}$ which is $\bar{\beta}$-ps-correct and therefore (by (b) for $\bar{\beta}$) belongs to $b_{\bar{\beta}}$; by induction $\bar{\bar{\beta}}$ is $\kappa_{\bar{\bar{\beta}}}, 1$-correct in $m_{\bar{\bar{\beta}}}$ and applying the embedding π from $m_{\bar{\bar{\beta}}}$ to $m_{\bar{\beta}}$ we conclude that $\pi(\bar{\bar{\beta}})$ is $\kappa_{\bar{\beta}}, 1$-correct in $m_{\bar{\beta}}$. As the critical point of π is at least $\bar{\bar{\beta}}$ it follows that $\bar{\bar{\beta}}$ is $\kappa_{\bar{\beta}}, 1$-correct in $m_{\bar{\beta}}$ as well. As $\bar{\beta}$ is a limit of such $\bar{\bar{\beta}}$, $\bar{\beta}$ is $\kappa_{\bar{\beta}}, 1$-correct in $m_{\bar{\beta}}$. If $\bar{\beta}$ is not a limit of strong limits then for a sufficiently large beth-number γ less than $\bar{\beta}$, the iteration along $b_{\bar{\beta}}$ carries some δ which is measurable and $\kappa_\gamma, 1$-correct in m_γ to $\bar{\beta}$ and therefore by elementarity $\bar{\beta}$ is $\kappa_{\bar{\beta}}, 1$-correct in $m_{\bar{\beta}}$.

For the second statement, if $\bar{\beta}$ is a limit of limits of $\bar{\beta}, n$-stables then we can apply induction to obtain cofinally-many $\bar{\beta}, n$-stable $\bar{\bar{\beta}}$ in $b_{\bar{\beta}}$ which are $\kappa_{\bar{\bar{\beta}}}, n+1$-correct in $m_{\bar{\bar{\beta}}}$; these $\bar{\bar{\beta}}$ are $\kappa_{\bar{\beta}}, n+1$-correct in $m_{\bar{\beta}}$ and therrefore so is $\bar{\beta}$, as it is a limit of such $\bar{\bar{\beta}}$. Otherwise for a sufficiently large $\bar{\beta}, n$-stable γ, the iteration along $b_{\bar{\beta}}$ carries some δ which is measurable and $\kappa_\gamma, n+1$-correct in m_γ to $\bar{\beta}$ and therefore by elementarity, $\bar{\beta}$ is $\kappa_{\bar{\beta}}, n+1$-correct in $m_{\bar{\beta}}$. $\Box$

It follows from Lemma 22 that for strong limits $\alpha < \beta$ and each n, α is nicely $\beta, n+1$-stable in V iff α is nicely $\beta, n+1$-strong or the limit of nicely $\beta, n+1$-strongs in m_0^*. As the strong limits of V are the measurable, $\mathrm{Ord}, 1$-corrects of m_0^* together with their limits, it follows that the Stability

Predicate is definable over m_0^*. This completes the proof of Theorem 4.

4. Further results

The Optimality of Mighty Mouse

Recall that δ is a *Woodin cardinal* if it is inaccessible and for each $A \subseteq \delta$, there is a $\kappa < \delta$ which is *A-strong below* δ; the latter means that for any cardinal $\alpha < \delta$ there is an elementary embedding j with critical point κ such that $j(\kappa) > \alpha$ and $A \cap \alpha = j(A) \cap \alpha$. Mighty Mouse can be characterised in terms of *definable Woodinness* as follows:

Proposition 23 *Let δ be inaccessible. The following are equivalent:*
(a) δ is definably-Woodin, i.e. satisfies the definition of being Woodin for $A \subseteq \delta$ which are definable over $H(\delta)$.
(b) δ is a limit of δ, n-strongs for each n.

Corollary 24 *Mighty Mouse is the least mouse with a measurable cardinal that is definably-Woodin.*

Proof of Proposition 23. Assuming (a), choose any $\alpha < \delta$ and let A be a subset of $\delta \setminus \alpha$ which codes all of $H(\delta)$; applying (a) to A yields a cardinal greater than α which is $\delta, 1$-strong. And for any n we can apply (a) to the pair of sets A, A_n where A_n is the $H(\delta)$-definable set of δ, n-strongs, yielding a $\delta, n+1$-strong greater than α. For (b) implies (a) it suffices to show that if κ is δ, n-strong then $H(\kappa)$ is Σ_{n+1}-elementary in $H(\delta)$, for then any $\delta, n+1$-strong κ witnesses Woodinness for subsets of $H(\delta)$ which are Σ_{n+1}-definable over $H(\delta)$ with parameters from $H(\kappa)$. We prove the former statement by induction on n. The base case $n = 0$ is clear since $H(\kappa)$ is Σ_1-elementary in V for any uncountable cardinal κ. Given the result for n, suppose that κ is $\delta, n+1$-strong and φ is a Σ_{n+2} sentence with parameters from $H(\kappa)$ which is true in $H(\delta)$. By induction and the unboundedness of the δ, n-strongs, we may choose a δ, n-strong λ so that φ is true in $H(\lambda)$. Applying the $\delta, n+1$-strength of κ we get an embedding $j : V \to M$ with critical point κ such that $\lambda < j(\kappa)$ is δ, n-strong in M. But then by elementarity, φ holds in some δ, n-strong of V less than κ, and therefore φ holds in $H(\kappa)$. $\square$

We also mention yet another useful equivalence of definable-Woodinness.

Lemma 25 *Suppose δ is inaccessible. δ is definably-Woodin iff for every function $f : \delta \to \delta$ that is definable over $H(\delta)$ there is an $\alpha < \delta$ closed under f which is f-*strong*; the latter means that there is an elementary embedding $j : V \to M$ with critical point α such that M contains $H(j(f)(\alpha))$.*

A proof of the above lemma can be found in [15] (where no definability is assumed; the same proof works with definability).

Lemma 26 *(Morse-Kelley) Assume that Ord is Mahlo*[28]*. Suppose that m is a mouse in which no cardinal is definably-Woodin and $(m_\gamma \mid \gamma \in Ord)$ is a definable iteration with total extenders of m. Then the resulting iteration tree has a definable branch of length Ord.*

Proof. For each inaccessible α choose a κ_α below α in the iteration tree order such that α is in the range of the corresponding map $\pi_{\kappa_\alpha,\alpha}$ from m_{κ_α} to m_α. Using the inaccessibility of α we can also require that κ_α be the critical point of $\pi_{\kappa_\alpha,\alpha}$ (and is therefore measurable in m_{κ_α}) and that $\pi_{\kappa_\alpha,\alpha}$ sends κ_α to α. By Fodor, there are stationary-many inaccessible α such that κ_α equals a fixed κ.

As κ is measurable in m_κ, it is not definably-Woodin in m_κ by hypothesis. Choose a function $f : \kappa \to \kappa$ which is definable over $m_\kappa|\kappa$ such that no $\bar{\kappa} < \kappa$ is f-strong in m_κ, or equivalently in the common part model M. Choose n so that f is Σ_n-definable over $m_\kappa|\kappa$. Let X be the definable stationary class of inaccessible α such that $\kappa_\alpha = \kappa$.

We argue that if $\alpha < \beta$ belong to X and are Ord, n-stable in the common part model M then α belongs to b_β = the set of iteration tree predecessors of β. If not, let $\bar{\kappa} < \alpha < \gamma + 1$ where $\bar{\kappa} < \gamma + 1$ are adjacent elements of b_β. Now $\bar{\kappa}$ is α-strong in M and therefore $\pi_{\kappa\beta}(f)(\bar{\kappa})$-strong in m_β, since $\pi_{\kappa\beta}(f)$ is Σ_n-definable over $M|\beta$ and α is Ord, n-stable in M. By elementarity there is an f-strong $\bar{\kappa} < \kappa$, contradicting the choice of f.

Thus we have a definable unbounded class Y such that if $\alpha < \beta$ belong to Y then α is below β on the iteration tree. This yields a definable branch through the iteration tree of length Ord. □

[28]Definable-Mahloness (i.e., the statement that every definable club contains an inaccessible) is sufficient.

Theorem 27 *(Morse-Kelley) Assume that Ord is Mahlo*[29] *and that M is the common part model of a definable Ord-iteration with total extenders of a mouse in which no measurable cardinal is definably-Woodin. Then V is not definably class-generic over M.*

Proof. By the preceding lemma let b be a definable branch through the iteration tree of length Ord. Let m_b be the direct limit of the m_α, $\alpha \in b$. As b is definable, m_b is wellfounded and there is some γ_0 such that some ordinal κ in m_{γ_0} is sent to Ord by the embedding from m_{γ_0} to m_b. Then the successive images of κ along b form a definable class I of indiscernibles for M. If $i < j$ belong to I then $M|i$ is elementary in $M|j$ and therefore each $M|i$, i in I, is elementary in M.

Now if V were class-generic over M via the M-definable forcing P, then in M we could define the class X of α such that some condition p in P forces α to belong to the definable class I. Note that if G is P-generic and contains the condition p then in $M[G]$, α belongs to the interpretation I^G of I according to G and therefore $M|\alpha$ is elementary in M. But then in M we can use the M-definable forcing relation for P (restricted to sentences of sufficient complexity) to define a satisfaction predicate for M, contradicting Tarski. □

Remark. The iteration tree of Section 3 cannot have a *definable* cofinal branch, else the above argument would imply that V cannot be definably class-generic over its common part model, contradicting what is shown in Secion 3.

Corollary 28 *(Morse-Kelley) Assume that Ord is Mahlo*[30]. *Then Mighty Mouse is the least mouse which captures V, in the following sense: Suppose that V is definably-generic over the common part model of a definable iteration of the mouse m. Then Mighty Mouse does not drop in its comparison with m.*

Proof. If there were such a drop then the mouse m would have no cardinal which is definably-Woodin in m. We may assume that the iteration of m does not drop and therefore by Theorem 27 we reach a contradiction. □

[29] Definable Mahloness suffices.

[30] Definable-Mahloness suffices.

Note that the satisfaction predicate for $(L[S], S)$ is not definable. Otherwise the argument of the final paragraph of the proof of Theorem 27 would show that V is not definably class-generic over this model.

Corollary 29 *Suppose that Mightty Mouse exists*[31] *and S_n denotes the Stability Predicate S restricted to n, i.e., the class of triples (α, β, k) in S with $k < n$. Then the satisfaction predicate for $(L[S_n], S_n)$ is definable.*

Proof. Iterate a proper initial segment of Mighty Mouse to capture S_n and not the full Stability Predicate S. The iteration tree has a definable branch of length Ord and yields a definable proper class of indiscernibles for the iterate. If α is such an indiscernible then $(L_\alpha[S_n], S_n \restriction \alpha)$ is fully elementary in $(L[S_n], S_n)$ and therefore satisfaction for the latter model is definable. $\square$

The Enriched Stable Core.

By enriching the Stable Core we obtain stronger forms of genericity over it. For our present purposes we work with a variant of the Enriched Stable Core of [8], defined as follows:

For strong limit $\alpha \leq \beta$ and $i <$ (β^+ of $L[H(\beta)]$) $\mathrm{SH}(\alpha, i)$ denotes the Skolem Hull of $H(\alpha)$ in $L_i[H(\beta)]$, i.e. the set of elements of $L_i[H(\beta)]$ which are definable in $L_i[H(\beta)]$ from parameters in $H(\alpha)$. We say that (β, i) is *suitable* if $\mathrm{SH}(\beta, i)$ is all of $L_i[H(\beta)]$. To suitable pairs (β, i) we associate the closed subset $C(\beta, i)$ of β consisting of all $\alpha < \beta$ such that $\mathrm{SH}(\alpha, i) \cap \beta = \alpha$. We say that β is *suitable* if there is a largest i, denoted i_β, so that $C(\beta, i)$ is unbounded in β. In this case we let $C(\beta)$ denote $C(\beta, i_\beta)$. We say that $\alpha < \beta$ is *β-stable* if $C(\alpha) = C(\beta) \cap \alpha$ and $(H(\alpha), C(\alpha)$ is Σ_1-elementary in $(H(\beta), C(\beta))$.

The *Enriched Stability Predicate* S^* is defined by:

$S^* = \{(\alpha, \beta, i) \mid \alpha < \beta$ are strong limit, β is suitable and α is β-stable$\}$

$(L[S^*], S^*)$ is the *Enriched Stable Core.*

[31] Actually it suffices to assume that mm^n exists, where mm^n denotes the least mouse with a measurable κ which is a limit of κ, n-strongs.

For any inner model (M, A), a class X is (M, A)-*constructible* if there is a wellfounded end-extension of M containing X and (M, A) as elements which satisfies "ZFC$^-$ + Every set is constructible from (M, A)".

Theorem 30 *([8], essentially) (Morse-Kelley) Let* $(L[S^*], \mathcal{C}^*)$ *be the model of Morse-Kelley where* $\mathcal{C}^*$ *consists of the* $(L[S^*], S^*)$*-constructible classes. Then* V *is generic over* $(L[S^*], \mathcal{C}^*)$ *for an* $(L[S^*], S^*)$*-definable, Ord-cc forcing.*

This is proved by forcing a predicate A which preserves the enriched stability predicate such that V is a definable inner model of $L[A]$ and, in analogy to the case of the ordinary Stable Core, showing that A is generic over $(L[S^*], \mathcal{C}^*)$ for a forcing built from infinitary quantifier-free sentences ordered under provability in the theory which expresses instances of the enriched stability predicate.

The Enriched Stable Core, like the Stable Core, can be strongly captured.

Definition 31 *Enriched Mighty Mouse, denoted emm, is the least mouse* m *with a measurable cardinal* κ *which in* m *is* constructibly-Woodin*, i.e. Woodin with respect to subsets of* κ *which are constructible from* $m|\kappa$*.*

Remark. emm exists if there is a mouse with a measurable cardinal above a Woodin cardinal. Thus the largest measurable of emm is only constructibly-Woodin, and not fully Woodin, in emm.

Theorem 32 *(Morse-Kelley) The Enriched Stability Predicate is definable over the common part model* M *of a definable Ord-iteration of Enriched Mighty Mouse.*

Proof. We define an iteration $((m_\gamma, E_\gamma) \mid \gamma < \mathrm{Ord})$ of $m_0 = \mathrm{emm} =$ Enriched Mighty Mouse as follows. We say that a pair (β, i) is *worrisome* (for m_γ) if, letting κ_γ denote the largest measurable of m_γ, one of the following holds:

1. β is less than κ_γ, β is suitable and some α less than β is not β-stable yet there is an extender E with critical point α which is $\beta, C^{m_\gamma}(\beta, i_\beta), 1$-strong in m_γ (i.e., E is $\beta, 1$-strong and $j_E(C^{m_\gamma}(\beta, i_\beta))$ agrees with $C^{m_\gamma}(\beta, i_\beta)$ below β).

2. $\beta = \kappa_\gamma$, (κ_γ, i) is suitable in m_γ and some $\alpha < \kappa_\gamma$ is measurable, $\kappa_\gamma, 1$-correct relative to $C^{m_\gamma}(\kappa_\gamma, i)$ and a limit of $\alpha, 1$-strongs relative to $C^{m_\gamma}(\kappa_\gamma, i) \cap \alpha$ in m_γ, yet α is not a limit of $\alpha, C^{m_\gamma}(\kappa_\gamma, i) \cap \alpha$, 1-stables in V.

Then we choose (β_γ, i_γ) to be lexicographically-least so that the pair (β_γ, i_γ) is worrisome and if Case 1 above holds, we choose E_γ to be the extender with least index witnessing Case 1 for (β_γ, i_γ). If Case 2 holds but Case 1 does not then we apply the order 0 measure on α where (i, α) is lexicographically-least witnessing Case 2. If there are no worrisome pairs then we apply the measure on κ_γ; the largest measurable of m_γ.

Now we take emm* to be the common part model of this iteration. Then the $\mathrm{Ord}, 1$-correct measurables of emm* together with their limits are the strong limits of V. For suitable β, if α is $\beta, C(\beta), 1$-strong below β then α is β-stable, by the worry-freeness of emm*. And conversely, as in the proof of Lemma 22(d), the β-*correct* α, i.e., the α so that $C(\alpha) = C(\beta) \cap \alpha$ and α is $\beta, 1$-correct relative to $C(\beta)$, lie on the branch b_β cofinal in β for suitable β; this yields the definability of b_β sufficient to verify that for suitable β, α is $\beta, 1$-strong relative to $C(\beta)$ in emm* if α is a successor β-stable in V. Note that by virtue of the genericity of $H(\beta)$ over $L_{i_\beta}[S^*|\beta]$ for a β-cc forcing that is definable without parameters over $(L_\beta[S^*], S^*|\beta)$, i_β as calculated in V is the same as i_β as calculated in emm* and the associated closed unbounded set $C(\beta)$ of V is the $C(\beta)$ of emm*. So the Enriched Stable Core is definable over emm*. $\square$

Corollary 33 *V is generic over the Morse-Kelley model $(M, \mathcal{C})$ for an Ord-cc forcing which is definable over M, where M is the common part model of a definable Ord-iteration of Enriched Mighty Mouse and $\mathcal{C}$ consists of the M-constructible subsets of M.*

Theorem 34 *(Morse-Kelley) Assume that Ord is Mahlo*[32]*. Suppose that V is Ord-cc generic over the Morse-Kelley model $(M, \mathcal{C})$ where M is a V-definable inner model and $\mathcal{C}$ consists of the M-constructible classes. Suppose that K_M = the core model of M exists. Then in the comparison of K_M with*

[32]Constructible-Mahloness (i.e., the statement that every V-constructible club contains an inaccessible) is sufficient.

Enriched Mighty Mouse, the latter does not drop. In particular, the reals of M contain the reals of Enriched Mighty Mouse.

Note that the above applies to $M =$ the Enriched Stable Core with $\mathcal{C} =$ the M-constructible classes and therefore:

Corollary 35 *Assume that Ord is Mahlo*[33]*. Then the reals of the Enriched Stable Core are the reals of Enriched Mighty Mouse.*

Proof of Theorem 34. Compare K_M with emm = Enriched Mighty Mouse, resulting in iteration trees $T(K_M)$, $T(\text{emm})$ of lengths λ_0, λ_1 (λ_0, $\lambda_1 \leq \text{Ord}$) and models $((K_M)_\gamma \mid \gamma < \lambda_0)$, $(\text{emm}_\gamma \mid \gamma < \lambda_1)$. Assume that there is a drop on the emm-side, which implies that $\lambda_1 = \text{Ord}$, and if $\lambda_0 < \text{Ord}$ then artificially pad the iteration on the K_M-side by declaring $\lambda_0 <_{T(K_M)} \gamma_0 <_{T(K_M)} \gamma_1$ for all ordinals $\lambda_0 < \gamma_0 < \gamma_1$, setting $(K_M)_\gamma = (K_M)_{\lambda_0}$ for $\gamma > \lambda_0$. Let K_M^* and emm* be the common part models on the two sides of the comparison.

As we are working in Morse-Kelley, we can define $L_i[\text{emm}^*]$ for each emm*-constructible wellorder i of Ord by iterating constructibility relative to emm* for i steps. And for such i we can define the associated $C(\text{Ord}, i)$, a closed unbounded subclass of Ord. Now as there is a drop on the emm-side, for some i, Ord is not a limit of $\text{Ord}, C(\text{Ord}, i)$-strongs in emm* and therefore as in Lemma 26, there is a cofinal branch $b(\text{emm})$ through $T(\text{emm})$ in $\mathcal{C}[V, K_M]$, the V, K_m-constructible = the V-constructible classes. $\mathcal{C}(V)$ (recall that M is V-definable). Similarly, there is a cofinal branch $b(K_M)$ through $T(K_M)$ which belongs to $\mathcal{C}[V]$.

On the K_M-side, no ordinal is sent to Ord by the map from $(K_M)_\gamma$ to $(K_M)_{\text{Ord}}$ = the direct limit of the $(K_M)_\gamma$, $\gamma \in b(K_M)$, so for a club of γ in $b(K_M) \cap b(\text{emm})$, γ is a closure point of the map from K_M to $(K_M)_\gamma$. For such γ, γ is in fact a fixed point of this map provided there is no stage $i < \gamma$ such that γ has $(K_M)_i$-cofinality equal to the critical point of the extender used at stage i on the K_M-side of the iteration. Even in this case we still get that γ^+ of K_M is at most γ^+ of $(K_M)_\gamma$ by the following general lemma.

[33] Constructible-Mahloness suffices.

Lemma 36 *Suppose that $j : M \to N$ is an elementary embedding and γ is an M-cardinal closed under j. Then γ^+ of N is at least γ^+ of M.*

Proof.[34] For α between γ and γ^+ of M choose a wellorder w_α of γ of length α. Let $f(\alpha)$ be the ordertype of $j(w_\alpha)|\gamma$. If α is less than β then $w_\alpha|\bar{\gamma}$ has ordertype less than that of $w_\beta|\bar{\gamma}$ for a club C of $\bar{\gamma} < \gamma$; it follows that f is order-preserving, as γ belongs to $j(C)$. So the ordertype of γ^+ of M is at most γ^+ of N. $\square$ (Lemma 36)

Since for a closure point γ of the embedding from K_M to $(K_M)_\gamma$ in $b(K_M)$ the iteration from $(K_M)_\gamma$ to $(K_M)_{\mathrm{Ord}}$ is above γ, all subsets of γ in $(K_M)_\gamma$ belong to $(K_M)_{\mathrm{Ord}}$ and therefore to emm* and to emm$_\gamma$. It follows that γ^+ of $(K_M)_\gamma$ is at most γ^+ of emm$_\gamma$ for such γ.

But the mouse emm$_\gamma$ belongs to $L_{\gamma^*}[H(\gamma)]$, where γ^* is least such that this structure is Σ_1-admissible and $b(K_M)|\gamma$ and $b(\mathrm{emm})|\gamma$ belong to it. Now we may assume that γ is chosen so that $H(\gamma)$ is generic over $(H(\gamma), L_{\gamma^*}[H(\gamma)] \cap V_{\gamma+1})^M$ and therefore γ^+ of $(K_M)_\gamma$, which is at most the height of emm$_\gamma$, is collapsed in M. By Lemma 36, γ^+ of K_M is also collapsed in M. As V is Ord-cc generic over $(M, \mathcal{C})$, the club of γ's as above contains a club in $\mathcal{C}$ and therefore we can choose γ as above to be M-singular, contradicting weak covering for M. $\square$

Using a similar argument we get:

Theorem 37 *(Morse-Kelley) Suppose that Ord is Mahlo*[35] *and $(M, \mathcal{C})$ is a model of Morse-Kelley where M is a definable inner model and $\mathcal{C}$ consists of the M-constructible classes. Suppose that the sound mouse m projects to ω and does not belong to M, and that Enriched Mighty Mouse drops in its comparison with m. Finally, suppose that K_M = the core model of M exists. Then m is not Ord-cc generic over $(M, \mathcal{C})$.*

Proof. As m is sound, projects to ω and does not belong to M, it follows that as emm drops in its comparison with m it also drops in its comparison with K_M. Now apply Theorem 34. $\square$

[34] I thank the anonymous referee for supplying this argument, an improvement on the one I gave for a weaker version of this Lemma (that was nevvertheless sufficient for the proof of Theorem 34).

[35] Constructible Mahloness suffices.

Enriched Mighty Mouse is minimal for Corollary 33, i.e. for obtaining the genericty of V over a Morse-Kelley model through mouse-iteration.

Corollary 38 *(Morse-Kelley) Suppose that Ord is Mahlo*[36] *and V is generic over $(M, \mathcal{C})$ for a forcing in $\mathcal{C}$ where M is the common part model of a definable Ord-iteration of the mouse m and $\mathcal{C}$ consists of the M-constructible subsets of M. Then Enriched Mighty Mouse does not drop in its comparison with m.*

Local Set-Genericity

It is also possible to show that V is generic over the truncation to Ord M of an Ord-iterate via a definable iteration of a mouse with every set in V being set-generic over M. This latter property is called *local set-genericity.*

Lemma 39 *Suppose that m_w is a mouse with a Woodin cardinal δ_w that is not a limit of Woodin cardinals. Suppose that δ is an inaccessible greater than the height of m_w. Then m_w has an iterate m_w^* which is definable with m_w as parameter such that δ_w^*, the image of δ_w under the iteration map, is a V-cardinal less than δ such that $H(\delta_w^*)$ is generic over m_w^* for a forcing which is definable over $m_w^*|\delta_w^*$ and δ_w^*-cc in m_w^*.*

Proof. This is like the proof of Theorem 32, but iterating only above the Woodin cardinals less than δ_w and working at stage γ of the iteration not with triples (α, β, i) with i less than β^+ of $L[H(\beta)]$, but with i less than β^+ of $m_\gamma[H(\beta)]$, where m_γ is the iterate produced at stage γ of the iteration. In the proof of Theorem 32 we iterated emm to ensure that stability is expessible in terms of strength, hitting the top measure at stages where our goal has been accomplished below it. In the present proof we stop at the first stage where our goal has been accomplished below the image of δ_w. The existence of an inaccessible δ above the height of m_w ensures that there is such a stage. $\square$

Remark. The previous lemma can also be proved using Woodin's genericity iterations, provided the requirement that the iterate m_w^* be definable with m_w as parameter is removed.

[36]Constructible-Mahloness suffices.

Lemma 40 *Suppose that $m^{\#}_{pcw}$ is the least mouse with a measurable limit of Woodin cardinals and there is a proper class of inaccessibles. Then there is a definable Ord-iteration of $m^{\#}_{pce}$ yielding an Ord-iterate $(m^{\#}_{pcw})^*$ such that every set in V is set-generic over $(m^{\#}_{pcw})^*|Ord$.*

Proof. This is an elaboration on the proof of Lemma 39. Note that in $m^{\#}_{pcw}$ there is no Woodin limit of Woodin cardinals. Now perform an iteration as in the previous proof to send the least Woodin cardinal of $m^{\#}_{pcw}$ to a cardinal δ_0 less than the least inaccessible, follow this with an iteration above δ_0 to send the second Woodin cardinal of this iterate to a cardinal δ_1 less than the next inaccessible and continue until all Woodin cardinals of the iterate have been moved, using the nonexistence of a measurable limit of of Woodin cardinals (below the top measurable) to avoid creating new Woodin cardinals in the iteration. Then apply the top measure of the iterate to create new Woodin cardinals and continue the iteration. After Ord steps the Woodin cardinals of the Ord-iterate $(m^{\#}_{pcw})^*$ of the iteration are the δ_i's and every set is set-generic over $(m^{\#}_{pcw})^*|\text{Ord}$ because $H(\delta_i)$ is generic over $(m^{\#}_{pcw})^*|\text{Ord}$ for a forcing definable over $(m^{\#}_{pcw})^*|\delta_i$ for each i. □

Theorem 41 *Suppose that mm_{pcw} is the least mouse with a measurable cardinal κ which is both a limit of κ, n-strongs for each n (i.e. is definably Woodin) and a limit of Woodin cardinals. Suppose there is a proper class of inaccessibles. Then there is a definable Ord-iterate mm^*_{pcw} of mm_{pcw} such that every set in V is set-generic over $mm^*_{pcw}|Ord$ and V is definably class-generic over $mm^*_{pcw}|Ord$.*

Proof. Here we combine the previous argument, moving Woodin cardinals, with the Mighty Mouse argument to get V generic over the truncation M at Ord of the Ord-iterate. At stage γ of the iteration we let E_γ witness a worrisome pair (β, i) for the purpose of ensuring that every set is set-generic over M, if there is such a pair, and if not, then we choose E_γ to witness a worrisome β for the purpose of ensuring that V is class-generic over M. At a definable club of stages there will be no worrisome pairs for the first purpose and therefore we will be able to ensure the expressibility in M of Ord, n-stability in terms of Ord, n-strength for each n. □

Non-rigidity of the Stable Core

Lemma 42 *Assume Mighty Mouse exists and there is a satisfaction predicate for V. Then the Stable Core is not rigid.*

Proof. Let M be the common part model of the iteration of mm used to strongly capture the Stable Core. Thus α is a measurable 1-correct or the limit of such in M iff α is strong limit in V, and α is nicely β, n-strong or a limit of such in M iff α is nicely β, n-stable in V (we refer to the conjunction of these properties as "stability equals strength").

Let $\pi : M \to \mathrm{Ult}(M, U) = M^*$ be the ultrapower of M by the measure U on the least measurable of mm. Note that even though U is not a member of M, M and mm have the same subsets of the least measurable of mm and therefore this ultrapower makes sense.

Lemma 43 *Supppose that α is nicely β, n-strong in M. Then α is also nicely β, n-strong in M^*.*

Proof of Lemma. Note that α is not moved by π as it is inaccessible in M. So by elementarity, α is nicely $\pi(\beta), n$-strong in M^*. We may assume that n is greater than 0 and therefore by assumption β is a limit of $\beta, n-1$-strongs in M. As β is strong limit it is closed under π and therefore by elementarity β is also a limit of $\pi(\beta), n-1$-strongs in M^*. It follows that α is nicely β, n-strong in M^*, as desired. $\square$ (Lemma)

Thus in M^* we have "stability implies strength", but the converse will not hold. (for example, there are measurable 1-corrects in M^* which are not strong limit). However just as in the iteration of mm to strongly capture the Stable Core, we can iterate M^* to achieve "stability equals strength", as the measurable 1-corrects and β, n-strong cardinals needed for this are available in M^*. Let M^{**} denote the common part model of this iteration. Then using the satisfaction predicate for V we can produce a cofinal branch through the iteration tree associated to this iteration and thereby an embedding $\pi^* : M^* \to M^{**}$. The composition $\pi^* \circ \pi : M \to M^{**}$ is an embedding which sends the Stable Core to itself, establishing this model's nonrigidity. $\square$

Some open questions

1. Is the Stable Core a generic extension of an iterate of Mighty Mouse? In particular, does it contain all the reals of Mighty Mouse? If V is definably-generic over an inner model M, must M contain all the reals of Mighty Mouse?

2. Assuming that Mighty Mouse exists, what large cardinals exist in the Stable Core? Does the Stable Core have arbitrarily large n-strong cardinals for each n, assuming there are such cardinals in V?

3. Suppose that Mighty Mouse does not belong to the model M. Is M the Stable Core of one of its generic extensions?

4. Suppose that every set is set-generic over the inner model M. Must M contain the reals of $m^{\#}_{pcw}$ (the least mouse with a limit κ of Woodin cardinals, with a measure on κ as an amenable predicate)? If in addition V is definably class-generic over M, must M contain the reals of mm_{pcw} (the least mouse with a κ which is both a limit of of Woodin cardinals and of κ, n-strongs for each n, with a measure on κ as an amenable predicate)?

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