

Internal consistency and the inner model hypothesis (Budapest lecture, August 2005)

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There are two standard ways to establish consistency in set theory. One is to prove consistency using *inner models*, in the way that Gödel proved the consistency of GCH using the inner model L . The other is to prove consistency using *outer models*, in the way that Cohen proved the consistency of the negation of CH by enlarging L to a forcing extension $L[G]$.

But we can demand more from the outer model method, and we illustrate this by examining Easton's strengthening of Cohen's result:

Theorem 1 (*Easton's Theorem*) *There is a forcing extension $L[G]$ of L in which GCH fails at every regular cardinal.*

Assume that the universe V of all sets is rich in the sense that it contains inner models with large cardinals. Then what is the relationship between Easton's model $L[G]$ and V ? In particular, are these models *compatible*, in the sense that they are inner models of a common third model? If not, then the failure of GCH at every regular cardinal is consistent only in a weak sense, as it can only hold in universes which are incompatible with the universe of all sets. Ideally, we would like $L[G]$ to not only be compatible with V , but to be an inner model of V .

We say that a statement is *internally consistent (relative to large cardinals)* iff it holds in some inner model (under the assumption that there are inner models with large cardinals). By specifying what large cardinals are required, we obtain a new type of consistency result. Let $\text{Con}(\text{ZFC} + \varphi)$

stand for “ZFC + φ is consistent” and $\text{Icon}(\text{ZFC} + \varphi)$ stand for “there is an inner model of ZFC + φ ”. A typical consistency result takes the form

$$\text{Con}(\text{ZFC} + \text{LC}) \rightarrow \text{Con}(\text{ZFC} + \varphi)$$

where LC denotes some large cardinal axiom. An *internal* consistency result takes the form

$$\text{Icon}(\text{ZFC} + \text{LC}) \rightarrow \text{Icon}(\text{ZFC} + \varphi).$$

Thus a statement φ is internally consistent relative to large cardinals iff $\text{Icon}(\text{ZFC} + \varphi)$ follows from $\text{Icon}(\text{ZFC} + \text{LC})$ for some large cardinal axiom LC.

A statement can be consistent without being internally consistent relative to large cardinals. . An example is the statement that there are no transitive models of ZFC, which fails in any inner model (assuming there are inner models with inaccessible cardinals). Another example is:

For each infinite regular cardinal κ there is a nonconstructible subset of κ whose proper initial segments are constructible.

This can be forced over L , but does not hold in any inner model, assuming the existence of $0^\#$.

If the consistency of a statement without parameters is shown using set forcing, then it is usually easy to prove its internal consistency relative to large cardinals. (Some examples are mentioned below.) But this is not the case for statements that contain uncountable parameters or for statements whose consistency is shown through the use of class forcing. In these latter cases, questions of internal consistency and of internal consistency strength can be quite interesting, as we shall now see.

Easton’s theorem revisited

Let Reg denote the class of infinite regular cardinals and Card the class of all infinite cardinals. An *Easton function* is a class function $F : \text{Reg} \rightarrow \text{Card}$ such that:

For all $\kappa \leq \lambda$ in Reg: $F(\kappa) \leq F(\lambda)$.

For all $\kappa \in \text{Reg}$: $\text{cof}(F(\kappa)) > \kappa$.

Easton showed that if F is an Easton function in L , then there is a cofinality-preserving class forcing extension $L[G]$ of L in which $2^\kappa = F(\kappa)$ for all regular κ . We say that the model $L[G]$ *realises* the Easton function F .

Which Easton functions in L can be realised in an inner model? The following are some partial results, obtained jointly with my student Pavel Ondrejović ([8]).

Theorem 2 *Suppose that $0^\#$ exists and F is an Easton function in L which is L -definable using parameters which are countable in V . Then there exists an inner model with the same cofinalities as L in which $2^\kappa = F(\kappa)$ for each infinite regular κ .*

Corollary 3 *The statement*

$$2^\kappa = \kappa^{++} \text{ for all infinite regular } \kappa$$

is internally consistent relative to $0^\#$.

Internal consistency relative to large cardinals can sometimes be obtained when we allow uncountable parameters.

Theorem 4 *Assume that $0^\#$ exists and κ is a regular uncountable cardinal. Then there is an inner model with the same cofinalities as L in which GCH fails only at κ .*

How badly can GCH fail in an inner model? The proof of the following uses the existence of a gap-1 morass in $L[0^\#]$.

Theorem 5 *Assume that $0^\#$ exists and κ is a regular uncountable cardinal. Then there is an inner model with the same cofinalities as L in which $2^\kappa = (\kappa^+)^V$.*

GCH fails below κ in Theorem 5. If we require that GCH hold below κ we obtain a weaker conclusion:

Theorem 6 *Assume that $0^\#$ exists, κ is a regular uncountable cardinal and α is less than $(\kappa^+)^V$. Then there is an inner model with the same cofinalities as L in which GCH holds below κ and $2^\kappa > \alpha$.*

Question. Suppose that $0^\#$ exists and κ is a regular cardinal. Is there necessarily an inner model where GCH holds below κ and $2^\kappa = (\kappa^+)^V$?

Conjecture. Assume the existence of $0^\#$. Then an L -definable Easton function F can be realised in an inner model M (having the same cofinalities as L) iff it satisfies:

$$F(\kappa) < (\kappa^{++})^V \text{ for all } \kappa \in \text{Reg}^L.$$

The singular cardinal hypothesis

The analog of Cohen's result for the singular cardinal hypothesis is:

Theorem 7 (*Gitik [10]*) *Suppose that K is an inner model satisfying GCH which contains a totally measurable cardinal κ , i.e., a cardinal κ of Mitchell order κ^{++} . Then there is a generic extension $K[G]$ of K in which κ is a singular strong limit cardinal and GCH fails at κ .*

By work of Mitchell [11], a totally measurable cardinal is necessary. Now consider the following weak analogue of Easton's result for the singular cardinal hypothesis:

(Global Gitik) GCH fails on a proper class of singular strong limit cardinals.

The proof of the previous theorem shows:

Theorem 8 *Suppose that K is an inner model satisfying GCH which contains a proper class of totally measurable cardinals. Then there is a generic extension $K[G]$ of K in which Global Gitik holds.*

Is Global Gitik internally consistent relative to large cardinals? In analogy to Easton's theorem, we might expect to show that the generic extension $K[G]$ of Theorem 8 can be obtained as an inner model. This is however not true for the natural choice of K , using covering arguments. The following work is joint with my student Tomáš Futáš ([7], to appear).

Theorem 9 *Suppose that there is a $\#$ for a proper class of totally measurable cardinals and let K be the “natural” inner model with a class of totally measurable cardinals. (K is obtained by taking the least iterable mouse m with a measurable limit of totally measurable cardinals and iterating its top measure to infinity.) Then there is no inner model of the form $K[G]$, where G is generic over K , in which Global Gitik holds.*

On the other hand, it is possible to choose K differently, so as to witness the internal consistency relative to large cardinals of Global Gitik:

Theorem 10 *Suppose that there is an inner model containing a measurable limit κ of totally measurable cardinals, where κ is countable in V . Then there is an inner model in which Global Gitik holds.*

This is proved as follows: Using the proof of Theorem 8, force over the given inner model to obtain a failure of the GCH on a set of singular strong limit cardinals cofinal in κ . This forcing preserves the measurability of κ . Using the countability of κ , the generic exists in V . Now iterate κ to infinity; the resulting model is a model of Global Gitik.

What is the internal consistency strength of Global Gitik, i.e., what large cardinal hypothesis must hold in some inner model to obtain an inner model of Global Gitik?

Theorem 10 provides an upper bound. In analogy to the proof of the internal consistency relative to $0^\#$ of Easton’s result, one would expect that a $\#$ for a proper class of totally measurables, a weaker assumption, would also suffice.

But unlike with Easton’s result, it is possible that the internal consistency strength of Global Gitik is the same as its (external) consistency strength, i.e., just a proper class of totally measurable cardinals, without its $\#$; the next result is an example of this unexpected phenomenon.

A cardinal κ is *Jonsson* iff every structure of cardinality κ for a countable language as a proper substructure of cardinality κ . By work of Mitchell [11], if there is a singular Jonsson cardinal then there is an inner model with a measurable cardinal. Conversely, if M is an inner model with a measurable

cardinal then $M[G]$ has a singular Jonsson cardinal, when G is Prikry generic over M . But in fact an inner model with a singular Jonsson cardinal can be obtained inside M :

Theorem 11 *Suppose that there is an inner model with a measurable cardinal. Then there is an inner model with a singular Jonsson cardinal.*

This is proved as follows: Let κ be measurable in an inner model M . Iterate M using the measure on κ to $M = M_0 \supseteq M_1 \supseteq \dots$, and let M^* be M_ω . Then $\langle \kappa_n \mid n \in \omega \rangle$ produces a Prikry generic G over M^* and $M^*[G]$ is an inner model with a singular Jonsson cardinal.

Thus the internal consistency strength of a singular Jonsson cardinal is the same as its consistency strength, that of one measurable cardinal. Is the situation similar with Gitik's Theorem 7? I.e., can Con be replaced with Icon in the implication $\text{Con}(\text{ZFC} + \text{there exists a totally measurable}) \rightarrow \text{Con}(\text{ZFC} + \text{GCH fails at a singular strong limit})$? Equivalently:

Question. Suppose that there is an inner model with a totally measurable cardinal. Then is there an inner model in which the GCH fails at a singular strong limit cardinal?

Two more internal consistency results

Katherine Thompson ([9]) and I have studied the global complexity of universal classes for certain types of structures. For a regular cardinal λ , we say that a poset P *omits λ chains* iff there is no order-preserving embedding of λ into P . For a regular cardinal $\kappa \geq \lambda$, let $O(\kappa, \lambda)$ denote the collection of posets of cardinality κ which omit λ chains. Then the complexity of $O(\kappa, \lambda)$, written $K(\kappa, \lambda)$, is the smallest cardinality of a subset S of $O(\kappa, \lambda)$ such that every element of $O(\kappa, \lambda)$ can be embedded into an element of S . Thompson and I obtained the following “high complexity” internal consistency result.

Theorem 12 *Assume that $0^\#$ exists. Suppose that F is an Easton function in L which is L -definable without parameters. Also suppose that λ is a parameter-free L -definable function which to each L -regular cardinal $\kappa > \omega$ associates a regular L -cardinal $\lambda(\kappa) \leq \kappa$. Then there is an inner model with the same cofinalities as L in which $2^\kappa = F(\kappa) = K(\kappa, \lambda(\kappa))$ for each L -regular $\kappa > \omega$.*

For F and λ as above, we also obtain the external consistency of $2^\kappa = F(\kappa)$ and $K(\kappa, \lambda(\kappa)) = \kappa^+$ for each L -regular $\kappa > \omega$ (“low complexity”). But we do not know if this statement is *internally* consistent relative to large cardinals.

The situation is similar concerning a joint result with Natasha Dobrinen ([2]). For cardinals $\kappa < \lambda$, κ regular and uncountable, let $P_\kappa(\lambda)$ denote the set of subsets of λ of cardinality less than κ . A result of Avraham-Shelah is that if G is generic over V for a ccc forcing that adds a real, then $P_\kappa(\lambda) \setminus V$ is stationary in $V[G]$ for all $\kappa < \lambda$. Dobrinen and I show:

Theorem 13 *It is consistent relative to the existence of a proper class of ω_1 -Erdős cardinals that GCH holds and $P_\kappa(\lambda) \setminus V$ is stationary in $V[G]$ for all regular κ greater than ω_1 , where G is generic over V for ω_1 -Cohen forcing.*

The property expressed in this theorem is internally consistent relative to a proper class of ω_1 -Erdős cardinals provided we restrict κ to be a successor cardinal; otherwise the question is open and would appear to require at least the internal consistency of a proper class of Woodin cardinals.

The inner model hypothesis

Recall that a statement is internally consistent iff it holds in some inner model. Therefore the meaning of internal consistency depends on what inner models exist. If we enlarge the universe, it is possible that more statements become internally consistent.

The *weak (or parameter-free) inner model hypothesis* asserts that the universe has been maximised with respect to internal consistency in the following sense: if a statement without parameters holds in an inner model of some outer model of V (i.e., in a model compatible with V) then it already holds in an inner model of V . This can be formalised by regarding V as a countable transitive model of ZFC, taking the countable transitive models of ZFC of the same ordinal height which contain it as its outer models and taking the countable transitive models of ZFC of the same ordinal height contained in it as its inner models.

The (strong) inner model hypothesis, introduced later, has considerable large cardinal strength. This shows that a considerable part of our basic assumption, that of the internal consistency of large cardinals, is derivable from a natural absoluteness principle, the maximisation of internal consistency.

We next observe that the weak inner model hypothesis can be regarded as a second-order generalisation of:

Parameter-free Lévy-Shoenfield absoluteness. Suppose that φ is a Σ_1 sentence true in an extension of V . Then φ is true in V .

Recall that Σ_1 formulas are *persistent* in the sense that if such a formula is true in a transitive set, it is also true in all larger transitive sets. We consider persistent *second-order* formulas: A formula is *persistently* Σ_1^1 iff it is of the form

$$\exists M(M \text{ is a transitive class and } M \models \psi),$$

where ψ is first-order. We now regard V as a model of (Gödel-Bernays) class theory, endowed with both sets and classes. By an *outer model of V* we mean a model of class theory V^* , with the same ordinals as V , whose sets include the sets of V and whose classes include the classes of V . Clearly if V satisfies a persistent Σ_1^1 formula then so do all of its outer models.

Theorem 14 *The following are equivalent:*

- (a) *(Parameter-free persistent Σ_1^1 absoluteness).* If a parameter-free Σ_1^1 formula is true in an outer model of V then it is true in V .
- (b) *(Weak inner model hypothesis).* If a first-order sentence is true in some model compatible with V then it is true in some inner model of V .

Proof. (a) \rightarrow (b): Suppose that the first-order sentence φ is true in some model M compatible with V . Then M is an inner model of some outer model V^* of V . The existence of M can be expressed as a parameter-free Σ_1^1 formula, and therefore by (a), φ holds in an inner model of V .

(b) \rightarrow (a): Suppose that the parameter-free persistent Σ_1^1 formula $\varphi \equiv \exists M(M \text{ is a transitive class and } M \models \psi)$ holds in some outer model V^* of V . Then ψ is true in some model compatible with V and therefore by hypothesis, ψ is true in some inner model of V . It follows that φ holds in V . \square

Remark. The weak inner model hypothesis implies absoluteness for sentences which are Σ_2 over $H(\omega_1)$ (equivalently, for sentences which are Σ_3^1 in the sense of descriptive set theory). This is because by Lévy-Shoenfield absoluteness, such a sentence is true iff it is true in some inner model.

Theorem 15 (a) *The weak inner model hypothesis implies that for some real R , ZFC fails in $L_\alpha[R]$ for all ordinals α . In particular, there are no inaccessible cardinals and the reals are not closed under $\#$.*
 (b) *The weak inner model hypothesis implies that $0^\#$ exists.*

Proof. (a) By a theorem of Beller-David (see [1]) there is an outer model V^* of V containing a real R such that $L_\alpha[R]$ fails to satisfy ZFC for each ordinal α . By the weak inner model hypothesis there is such a real R in V , as this property of R is absolute to any inner model containing R .

(b) By [3], if $0^\#$ does not exist then there is a sentence which is Σ_3^1 in the sense of descriptive set theory which is false in V but true in an outer model. Using the *Remark* above, this contradicts the weak inner model hypothesis. \square

Remark. A stronger conclusion could have been stated in Theorem 15 (b). The weak inner model hypothesis implies the existence of $0^\#, 0^{\#\#}, \dots, 0^{\#\alpha}, \dots$, as long as these reals are Solovay singletons, i.e., unique solutions to formulas of the form $L[x] \models \varphi(x)$.

Absolute parameters and the strong inner model hypothesis

How can we introduce parameters into the inner model hypothesis? The following result shows that inconsistencies arise without strong restrictions on the type of parameters allowed.

Proposition 16 *The inner model hypothesis with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.*

Proof. With arbitrary ordinal parameters, inconsistency results from the fact that \aleph_1 can be countable in an outer model. To obtain an inconsistency with arbitrary real parameters, argue as follows. By (a) of Theorem 15, even the parameter-free version implies the existence of a real R such that ω_1 equals

ω_1 of $L[R]$. Then the statement “ ω_1 of $L[R]$ is countable” (with parameter R) holds in some outer model but not in any inner model. \square

So instead we consider *absolute parameters*, as in [6]. For a set p and two transitive models V_0, V_1 containing p as an element, we say that p is *absolute between V_0 and V_1 via the formula ψ* iff ψ is a first-order formula without parameters which defines p both in V_0 and in V_1 .

Inner model hypothesis with arbitrary absolute parameters Suppose that p is absolute between V and V^* , where V^* is an outer model of V , and φ is a first-order sentence with parameter p which holds in an inner model of V^* . Then φ holds in an inner model of V .

For a singular cardinal κ , a \square_κ *sequence* is a sequence of the form $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$ such that each C_α has ordertype less than κ and for $\bar{\alpha}$ in $\text{Lim } C_\alpha$, $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$. *Definable \square_κ* is the assertion that there exists a \square_κ sequence which is definable over $H(\kappa^+)$ with parameter κ . We will be interested in the special case $\kappa = \beth_\omega$ (where $\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$ and $\beth_\omega = \bigcup_n \beth_n$), in which case the parameter κ is superfluous.

Theorem 17 *The inner model hypothesis with arbitrary absolute parameters is inconsistent.*

Proof. We first show that definable \square_κ fails, where κ is \beth_ω . Let $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$ be a \square_κ sequence definable over $H(\kappa^+)$ without parameters. For each n let S_n consist of all limit $\alpha < \kappa^+$ such that the ordertype of C_α is greater than \beth_n .

Claim. Let P_n be the forcing that adds a CUB subset of S_n using closed bounded subsets of S_n as conditions, ordered by end extension. Then P_n is κ^+ distributive, i.e., does not add κ -sequences.

Proof of Claim. It is enough to show that P_n is \beth_m^+ distributive for each $m < \omega$. Assume that m is greater than n . Suppose that p is a condition and $\langle D_i \mid i < \beth_m^+ \rangle$ are dense. Let $\langle M_i \mid i < \kappa^+ \rangle$ be a chain of size κ elementary submodels of some large $H(\theta)$ such that M_0 contains $\kappa \cup \{ \langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle, p \}$ and for each $i < \kappa^+$, $\langle M_j \mid j \leq i \rangle$ is an element of M_{i+1} . Let κ_i be $M_i \cap \kappa^+$ and C the set of such κ_i 's. Then a final segment D of $C \cap \text{Lim } C_\gamma$

is contained in S_n , where $\gamma = \kappa_{\beth_m^+}$. Write D as $\langle \kappa_{\alpha_i} \mid i < \text{ordertype } D \rangle$. We can then choose a descending sequence $\langle p_i \mid i < \beth_m^+ \rangle$ of conditions below p such that p_{i+1} meets D_i and belongs to $M_{\kappa_{\alpha_i+1}}$ for each i . Then the greatest lower bound of this sequence meets each D_i . This proves the Claim.

It follows that for each n the forcing P_n does not alter $H(\kappa^+)$. By the inner model hypothesis with absolute parameters S_n has a CUB subset C_n in V for each n . But this is a contradiction, as the intersection of the C_n 's is empty.

Now we refine the above argument. As not every real has a $\#$, there exist reals R such that κ^+ equals κ^+ of $L[R]$, where κ is \beth_ω . Let X be the set of such reals and for each R in X let $\langle C_\alpha^R \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$ be the $L[R]$ -least \square_κ sequence. Now for limit $\alpha < \kappa^+$, define C_α^* to be the intersection of the C_α^R , $R \in X$. Then $\langle C_\alpha^* \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$ is definable in $H(\kappa^+)$ without parameters and has the properties of a \square_κ sequence with the sole exception that C_α^* is only guaranteed to be unbounded in α if α has cofinality greater than 2^{\aleph_0} . Now repeat the above argument using $\langle C_\alpha^* \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$ in place of $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$, to obtain a contradiction. \square

To obtain the (strong) inner model hypothesis, we restrict ourselves to absolute *ordinal* parameters.

Inner model hypothesis Suppose that the ordinal α is absolute between V and V^* , where V^* is an outer model of V , and φ is a first-order sentence with parameter α which holds in an inner model of V^* . Then φ holds in an inner model of V .

Remark. By strengthening the sentence φ , we may also require in the conclusion of the inner model hypothesis that p be absolute between V and an inner model of V witnessing φ , via the same formula witnessing the absoluteness of p between V and V^* .

Theorem 18 *The inner model hypothesis implies the existence of an inner model with a strong cardinal.*

Proof. Let K be the core model below a strong cardinal (see [12]). Let κ be \beth_ω and $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$ the least \square_κ sequence of K ; this

is also a \square_κ sequence in V , else κ^+ is greater than κ^+ of K , yielding an inner model with a strong cardinal. As in the proof of Theorem 17, there are generic extensions of V preserving $H(\kappa^+)$ which add CUB subsets to each $S_n = \{\alpha < \kappa^+ \mid \text{ordertype } C_\alpha > \beth_n\}$. It follows from the inner model hypothesis (and the previous Remark) that for each n there is an inner model M_n with the correct \beth_ω^+ in which $S_n^{M_n}$ contains a CUB subset C_n , where $S_n^{M_n}$ is defined using the least \square_κ sequence of K^{M_n} (which may differ from K). However as κ^+ equals κ^+ of M_n and therefore equals κ^+ of K^{M_n} , it follows that the $K_{\kappa^+}^{M_n}$'s compare to a common K^* of height κ^+ , fixing ordinals of sufficiently large cofinality in some CUB subset C of κ^+ . Therefore the intersection of the $S_n^{K^*}$'s, which is empty, contains all ordinals of sufficiently large cofinality in some CUB subset of κ^+ , a contradiction. \square

Remark. It is commonly conjectured that core model theory can be extended from strong cardinals to Woodin cardinals, without any large cardinal assumptions. If this is the case, then the inner model hypothesis implies the existence of an inner model with a Woodin cardinal.

Question. Is the (weak or strong) inner model hypothesis consistent relative to large cardinals? If so, what is its consistency strength?

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