Combinatorics and Large Cardinals

Combinatorial Set Theory = Infinitary Combinatorics

Special case: *Large Cardinal Combinatorics*

Some Topics:

Large cardinal arithmetic
Large cardinal characteristics
Graphs and large cardinals
Trees and large cardinals
(Partition relations and large cardinals) : See Foreman-Hajnal and Džamonja-Larson-Mitchell

Large cardinals and L-like principles: $\diamondsuit$, $\Box$, Morass, Condensation

Large cardinal combinatorics and definability

Combinatorics and Large Cardinals

The General Approach

1. (Preparing) Start with a large cardinal κ:
$\kappa =$ critical point of a nice $j : V \to M$, definable in V

2. (Forcing) Obtain a combinatorial property of κ in a forcing extension $V[G]$, G P-generic

3. (Lifting) Verify that κ is still measurable in $V[G]$:
Lift $j : V \to M$ to $j^* : V[G] \to M[G^*]$, j^* definable in $V[G]$

In large cardinal combinatorics, forcing and elementary embeddings interact

More about Lifting

Lift $j : V \to M$ to $j^* : V[G] \to M[G^*]$, j^* definable in $V[G]$

Easy case: G adds no new κ-sequences ($V^\kappa \cap V[G] \subseteq V$)

Then (assuming a good Preparation) G^* is the $P^* = j(P)$-generic generated by $j[G]$

Harder case: Suppose that A is a new subset of κ added by G
Then $A = j^*(A) \cap \kappa$ belongs to $M[G^*]$
As $M \subseteq V$, G^* adds a new bounded subset of $j(\kappa)$
So G must add a new bounded subset of κ!

Example: To kill GCH at a measurable κ, P must not only add new subsets of κ but also new *bounded* subsets of κ

More about Lifting (continued)

Lift $j : V \to M$ to $j^* : V[G] \to M[G^*]$, j^* definable in $V[G]$

P should be an iteration $P = (P_\alpha \mid \alpha \leq \kappa)$
P-generic $G = (G_\alpha \mid \alpha \leq \kappa)$
$G(\alpha)$ adds new subsets of α (for many $\alpha < \kappa$)

So we have:

$j^* : V[G] = V[G(<\kappa)][G(\kappa)] \to$

$M[G^*(<\kappa)][G^*(\kappa)][G^*(\kappa, j(\kappa)][G^*(j(\kappa))]$

What is G^*?

Combinatorics and Large Cardinals

More about Lifting (continued)

$j^* : V[G] = V[G(<\kappa)][G(\kappa)] \to$
$M[G^*(<\kappa)][G^*(\kappa)][G^*(\kappa, j(\kappa)][G^*(j(\kappa))]$

Below κ: $G^*(<\kappa) = G(<\kappa)$
At κ: $G^*(\kappa) \approx G(\kappa)$
Filling the gap: $G^*(\kappa, j(\kappa))$ must be "built"
Last lifting: $G^*(j(\kappa)) \supseteq j[G(\kappa)]$

Below κ: Trivial
At κ: Sometimes nontrivial, but never difficult
Filling the gap: Usually trivial, but sometimes difficult
Last lifting: Difficult if $P(\alpha)$ uses α-Cohen,
but can be reduced to the "Easy case" if $P(\alpha)$ uses forcings with *good α-fusion* (e.g. α-Sacks, α-Miller)

Large Cardinal Arithmetic

Cohen: $2^{\aleph_0} = \aleph_2$
Easton: $2^\kappa = F(\kappa)$ for any Easton function F

Large cardinal versions of these results:

κ is *(κ^{++}-)hypermeasurable* iff κ is the critical point of $j : V \to M$ with $H(\kappa^{++}) \subseteq M$
κ is *totally measurable* iff κ has *Mitchell order* κ^{++}

Large Cardinal Cohen
Woodin: From κ hypermeasurable can force $2^\kappa = \kappa^{++}$, keeping κ measurable
(Uses α-Cohen. Last Lifting is difficult)
Gitik: Same, from κ totally measurable
(Uses α-Cohen. Both Preparation and Last Lifting are difficult)

Large Cardinal Arithmetic

Large Cardinal Easton (with Radek Honzík)

Global Woodin: For a *uniform* Easton function F, can force $2^\kappa = F(\kappa)$ for all regular κ, keeping κ measurable whenever it is $F(\kappa)$-hypermeasurable in the ground model
(Uses α-Sacks at inaccessible, α-Cohen elsewhere. Lifting at κ is nontrivial, Filling the Gap is difficult.)

Global Gitik: For $F(\kappa) = \kappa^{+n}$, n finite
(Like Global Woodin, but with tricks of Avraham to Fill the Gap.)

Large Cardinal Characteristics

The Cardinal Characteristic $\mathfrak{d}(\kappa)$ = *Dominating Number at* κ

The Easton result is:

(Cummings-Shelah): Can force $\mathfrak{d}(\alpha) = \alpha^+ < 2^\alpha$ for all regular α

(Uses α-Cohen and α-Hechler forcings)

The large cardinal result:

(with Katie Thompson) If κ is hypermeasurable then can force $\mathfrak{d}(\alpha) = \alpha^+ < 2^\alpha$ for all regular α, keeping κ measurable

(Uses α-Sacks, α-Cohen and α-Hechler forcings. The difficulty is in Filling the Gap.)

Graphs and Large Cardinals

Embedding Complexity for Graphs

$\alpha \leq \kappa$ infinite and regular

$G(\alpha, \kappa)$ = Graphs of size κ which omit α-cliques

Embedding complexity of $G(\alpha, \kappa) = \mathrm{ECG}(\alpha, \kappa)$:
Smallest size of a $U \subseteq G(\alpha, \kappa)$ such that every graph in $G(\alpha, \kappa)$ embeds into some element of U (as a subgraph)

Easton result: What are the possibilities for $\mathrm{ECG}(\alpha, \kappa)$ as a function of α and κ?

Graphs and Large Cardinals

Complexity triple (a, c, F):
$a, c, F : \mathrm{Reg} \to \mathrm{Card}$
F is an Easton function
$a(\kappa) \leq \kappa < c(\kappa) \leq F(\kappa)$ for all κ

(with Mirna Džamonja and Katie Thompson) If GCH holds and (a, c, F) is a definable complexity triple then can force $\mathrm{ECG}(a(\kappa), \kappa) = c(\kappa)$ and $2^\kappa = F(\kappa)$ for all regular κ

Large cardinal result? Open question.
Positive evidence is the following "internal consistency" result:

(with Katie Thompson) If (a, c, F) is an L-definable complexity triple then there is an inner model of $L[0^\#]$ with the same cofinalities as L in which $\mathrm{ECG}(a(\kappa), \kappa) = c(\kappa)$ and $2^\kappa = F(\kappa)$ for all regular κ

Trees and Large Cardinals

Mitchell: Starting with $\kappa < \lambda$ with κ regular and λ weak compact, can force the tree property at κ^{++}.

Proof uses "Mitchell forcing"

If κ is measurable then the tree property will hold at κ (as it holds at all weak compacts) and fail at κ^{+} (as it fails at successors of inaccessibles)

(with Natasha Dobrinen) If κ is λ-hypermeasurable where $\lambda > \kappa$ is weak compact then can force the tree property at κ^{++}, preserving the measurability of κ

Uses iterations of α-Sacks forcing. The work is in reducing the Last Lifting to the Easy case (more about this later). Related results:

Trees and Singular Cardinals

(with Halilović) If κ is λ-hypermeasurable where $\lambda > \kappa$ is weak compact then can force the tree property at $\aleph_{\omega+2}$, with $\aleph_\omega$ strong limit.

Similar techniques appear to give the tree property at all $\aleph_{2n}$, $0 < n < \omega$, and at $\aleph_{\omega+2}$ simultaneously, assuming slightly more than λ-hypermeasurability (must be checked).

One more Large Cardinal Characteristic

The Card Characteristic CofSym(α)

Sym(α) = group of permutations of α under composition
CofSym(α) = least λ such that Sym(α) is the union of a strictly increasing λ-chain of subgroups

(Sharp and Thomas) CofSym(α) can be any regular above α

A large cardinal result:

(with Lyubomyr Zdomskyy) If κ is hypermeasurable then can force CofSym(κ) = κ^{++}, keeping κ measurable

Uses iterations of α-Miller with continuous club-splitting and generalisations of α-Sacks. Difficulties are in the forcing part, which uses $\mathfrak{g}_{cl}(\kappa)$, the groupwise density number for *continuous* partitions, and in reducing the Last Lifting to the Easy case.

About the Last Lifting: Good Fusion

In the cases of the tree property and CofSym we used iterations of κ-Sacks and κ-Miller forcings.

In fact, any iteration of forcings with "good κ-Fusion" (not defined here) or with κ^+-strategic closure would work:

(with Radek Honzík and Lyubomyr Zdomskyy) If $j : V \to M$ is an ultrapower embedding witnessing hypermeasurability then there is a κ-cc "preparatory forcing" R of size κ such that if R forces P to be a κ^{++}-iteration of forcings either with good κ-Fusion or κ^+-strategic closure then $R * P$ preserves cardinals up to κ^+ and j lifts to $j^* : V[G] \to M[G^*]$ definably in $V[G]$ for any G which is $R * P$ generic over V. In particular, $R * P$ preserves the measurability of κ.

Related to Roslanowski-Shelah: Reasonably Bounded Forcings

Large Cardinals and L-like Principles

Assume GCH. We look at the compatibility of large cardinals with combinatorial principles that hold in L.

$\diamondsuit$ provably holds at measurable cardinals (indeed at all subtle cardinals).

$\square$ is more interesting.

κ is α^+-*subcompact* iff for each $A \subseteq H(\alpha^+)$ there are $\bar{\alpha} < \alpha$, $\bar{A} \subseteq H(\bar{\alpha}^+)$ and $\pi : (H(\bar{\alpha}^+), \bar{A}) \to (H(\alpha^+), A)$ sending its critical point to κ

(Jensen) If κ is κ^+-subcompact then $\square_\kappa$ fails

More generally:
(with Andrew Brooke-Taylor)
If κ is α^+-subcompact, $\kappa \leq \alpha$, then $\square_\alpha$ fails

Large Cardinals and L-like Principles

The failure of $\square_\kappa$ follows from *stationary reflection for* κ^+ *(on cofinality* ω*)*:

$\mathrm{SR}(\kappa^+, \omega)$: If $S \subseteq \kappa^+ \cap \mathrm{Cof}(\omega)$ is stationary then $S \cap \lambda$ is stationary for some $\lambda < \kappa^+$

and stationary reflection follows from a slight strengthening of subcompactness:

κ is α^+*-stationary subcompact* iff for each $A \subseteq H(\alpha^+)$ and stationary $S \subseteq \alpha^+$ there are $\bar{\alpha} < \alpha$, $\bar{A} \subseteq H(\bar{\alpha}^+)$, $\bar{S} \subseteq \bar{\alpha}^+$ and $\pi : (H(\bar{\alpha}^+), \bar{A}, \bar{S}) \to (H(\alpha^+), A, S)$ sending its critical point to κ such that $\bar{S}$ is stationary

(with Andrew Brooke-Taylor)
If κ is α^+-stationary subcompact then $\mathrm{SR}(\alpha^+, \omega)$ holds

Large Cardinals and L-like Principles

Our previous results are optimal:

(with Andrew Brooke-Taylor) Assume GCH and define:
$I = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \text{ is } \alpha^+\text{-subcompact})\}$ and
$J = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \text{ is } \alpha^+\text{-stationary subcompact})\}$.
Then there is a cofinality- and ZFC-preserving forcing P such that for P-generic G the following hold:

- $I^{V[G]} = I$ and $J^{V[G]} = J$
- $\mathrm{SR}(\alpha^+, \omega)$ fails in $V[G]$ for all $\alpha \notin J$
- $\Box_\alpha$ holds in $V[G]$ for all $\alpha \notin I$
- ω-superstrongs (extremely large cardinals) are preserved

$P =$ the iteration where: At stages $\alpha \in J$ do nothing, at stages $\alpha \in I \setminus J$ kill $\mathrm{SR}(\alpha^+, \omega)$ and at stages $\alpha \notin I$ add a witness to $\mathrm{SR}(\alpha^+, \omega)$, kill its stationarity and then force $\Box_\alpha$

Large Cardinals and L-like Principles

Andrew and I also show that one can force universal Gap 1 morasses at each regular cardinal, preserving ω-superstrongs

I briefly mention work on *Condensation Principles*

The L-hierarchy obeys *strong (or club) condensation*: Any elementary submodel of an L_α is isomorphic to some $L_{\bar{\alpha}}$.
But this form of condensation contradicts the existence of ω_1-Erdős cardinals.

One can weaken strong condensation to *stationary condensation*, but this is not very useful (it is too weak).

A better notion is *local club condensation*.
Adapting an argument of Neeman, Peter Holy shows:

Suppose that V carries an "acceptable" hierarchy witnessing local club condensation and satisfies $\square$ at small cofinalities. Then if PFA for $\mathfrak{c}^+$-linked forcings holds in a proper forcing extension there must be many subcompact cardinals in V.

(with Peter Holy) One can force the above hypothesis together with the existence of a single subcompact.
Therefore it is consistent that there is a subcompact yet PFA for $\mathfrak{c}^+$-linked forcings fails in all proper forcing extensions.

In other words, a subcompact is a "quasi lower bound" on the consistency strength of PFA for $\mathfrak{c}^+$-linked forcings. Viale-Weiß have a related result for the full PFA and "standard iterations".

Large Cardinal Combinatorics and Definability

We close by addressing the following question:

Can the objects that arise in large cardinal combinatorics be constructed definably?

For a large cardinal κ, such objects are typically subsets of $H(\kappa^+)$, so it suffices to obtain definable wellorders of $H(\kappa^+)$ in conjunction with large cardinal combinatorial properties.

(with David Asperó) Preserving very large cardinals it is possible to force a definable wellorder of $H(\kappa^+)$ for all regular uncountable κ. (Uses "strongly type-guessing" club sequences)

Large Cardinal Combinatorics and Definability

(with Radek Honzík) Starting with a hypermeasurable κ it is possible to keep κ measurable, force $2^\kappa = \kappa^{++}$ and add a definable wellorder of $H(\kappa^+)$.

(Remark: This can be used to obtain a definable failure of the Singular Cardinal Hypothesis)

The previous result begins with a model of GCH. However:

(with Phillip Lücke) If κ is supercompact then there is a κ^+-cc forcing which preserves supercompactness and adds a wellorder of $H(\kappa^+)$ which is Σ_1-definable with parameters.
(Uses "Kurepa tree coding")

Questions concerning the definability of large cardinal analogues of MAD families of sets and functions, maximal cofinitary groups, dominating or splitting families, etc. remain to be investigated.

HAPPY BIRTHDAY, ANDRÁS!