TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 255, November 1979

β-RECURSION THEORY[1]

BY

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ABSTRACT. We define recursion theory on arbitrary limit ordinals using the J-hierarchy for L. This generalizes α-recursion theory, where the ordinal is assumed to be Σ_1-admissible. The notion of *tameness* for a recursively enumerable set is defined and the degrees of tame r.e. sets are studied. Post's Problem is solved when $\Sigma_1 \mathrm{cf}\ \beta \geqslant \beta^*$. Lastly, simple sets are constructed for all β with the aid of a β-recursive version of Fodor's Theorem.

Introduction. Recursion theory was generalized from the integers to ω_1^{CK} = the first nonrecursive ordinal by Kreisel and Sacks **[8]** and then to an arbitrary Σ_1-admissible ordinal α by Kripke and Platek **[9]**, **[14]**. Since then, the subject of α-recursion theory, or recursion theory on the Σ_1-admissible ordinals, has flourished, generalizing theorems of ordinary recursion theory to many or sometimes all Σ_1-admissible α.

The key step toward accomplishing this program was taken by Sacks and Simpson **[17]** when they adapted the finite injury method, invented by Friedberg **[2]** and Muchnik **[13]** to solve Post's Problem in ordinary recursion theory, to arbitrary Σ_1-admissible ordinals. Sacks and Simpson used Skolem Hulls in the way that Gödel used them **[6]** to prove the GCH in L. The connection between α-recursion theory and set theory is partly explained by the fact that the α-r.e. sets are just the sets Σ_1-definable over L_α, the αth level of L.

This suggested that techniques used to analyze the structure of L would prove useful in α-recursion theory.

Ronald Jensen, in his fine structure theory, has greatly extended Gödel's ideas to provide a very deep analysis of how sets are constructed at each level of L, and has used this to settle many important questions of model theory and set theory in this model.

There is an asymmetry between the approach of α-recursion theory and

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Presented to the Society, November 4, 1975 under the title *Recursion theory without admissibility*; received by the editors January 11, 1977 and, in revised form, January 6, 1978.

AMS (MOS) subject classifications (1970). Primary 02F27; Secondary 02F25, 04A20.

Key words and phrases. β-recursion theory, priority arguments, fine structure of L, Σ_1-cofinality.

[1]This paper is extracted from the author's doctoral dissertatation, written at M.I.T. under the direction of Professor Gerald E. Sacks. We thank Professor Sacks for his advice and encouragement.

[2]We also thank the referee for his valuable suggestions.

that of Jensen's: α-recursion theory studies only the Σ_1-admissible levels of L whereas Jensen's work applies to all levels. The reason is that in α-recursion theory the essential problem is always to obtain a bound on an inductive process, which *appears* to require the assumption of admissibility (which is a bounding principle).

However, we feel that the main thrust of the work in α-recursion theory has been to demonstrate that recursion-theoretic constructions from ordinary recursion theory which seem to require a large amount of replacement, say Σ_2 or even Σ_3, can actually be refined to succeed with only the assumption of Σ_1-admissibility. In view of the applicability of Jensen's ideas to every level of L, it is natural to bring things to their logical conclusion and ask:

(*) Can the assumption of Σ_1-admissiblity be eliminated?

If one can make constructions which appear to involve Σ_2 of Σ_3 replacement succeed on an arbitrary Σ_1-admissible ordinal, can one try even harder and get by with no admissibility assumption?

β-recursion theory is the study of recursion theory on arbitrary limit ordinals. β-r.e., β-finite, and β-recursive sets are defined as they are in α-recursion theory. The passage from α-recursion theory to β-recursion theory is analogous to that from ordinary recursion theory to α-recursion theory. Firstly, distinctions appear which were not present in the less general theory: In α-recursion theory, the distinction appears between regular and nonregular sets (or hyperregular and nonhyperregular sets), though of course every set in ordinary recursion theory is both regular and hyperregular. In β-recursion theory, an important distinction appears between those β-r.e. sets which have "tame" enumerations (defined in Chapter 2) and those which do not, though every such set in α-recursion theory has a "tame" enumeration. Secondly, certain results do not generalize completely to the wider context: In α-recursion theory, the maximal sets theorem generalizes only to some admissible α (see **[10]**). In β-recursion theory, the regular sets theorem generalizes to only some inadmissible β (for some β-r.e. sets).

In both of these instances, the more general theory helps both to clarify the concepts of and to indicate what assumptions are necessary in the more specific one.

A positive answer to (*) would have great importance for α-recursion theory, as the unsolved problems in this subject result from lack of Σ_2-admissibility; if in fact admissibility is not necessary, then the techniques which demonstrate that should yield constructions which work for arbitrary α (even inadmissible α). This has actually been partially carried out in **[4]**, where Post's Problem is solved for many inadmissible β. What is used to make up for the lack of admissibility is, as predicted above, Jensen's fine structure theory.

The development of β-recursion theory has come in two parts This paper deals primarily with the first part, where properties of β-r.e. sets which are present with admissibility are explored in the context of inadmissible β. This resulted from joint efforts of the author and G. Sacks (see **[5]**) to narrow the collection of β-r.e. sets in order to develop a theory with some similarity to the admissible case. Results of Maass **[12]** and the author indicate that this yields a good theory when $\Sigma_1\text{cf}\ \beta \geqslant \beta^*$ and otherwise not. The second part of the development instead discards attempts to make β-recursion theory look like α-recursion theory and directly deals with severe failures of admissibility (Σ_1 cf $\beta < \beta^*$). This will be treated in the forthcoming **[4]**. This split into cases was first made evident by Jensen in his proof of Σ_2-Uniformization for S_β (see **[7]**).

It is our hope that Jensen's ideas and those from recursion theory on the ordinals will combine not only to solve many problems in generalized recursion theory, but also to provide an ultimately fine analysis of the structure of L.

In Chapter 1, the basic facts about the J-hierarchy as developed by Jensen in [7] are summarized and various types of cofinalities and projecta are defined. In Chapter 2, we present the key definitions of β-recursion theory. Chapter 3 establishes some basic results about the degrees of tamely r.e. sets: a regular sets theorem, a recursive upper bound to the t.r.e. degrees inadmissible β, and a solution to Post's Problem when $\Sigma_1\text{cf}\ \beta \geqslant \beta^*$. Chapter 4 uses a β-recursive version of Fodor's Theorem from combinatorial set theory to construct simple β-r.e. sets for all β and to characterize those β for which there is a non-β-finite t.r.e. subset of β^*.

CHAPTER 1. THE FINE STRUCTURE OF L

We begin by describing a hierarchy for Gödel's L which differs somewhat from the usual one. This hierarchy is due to Ronald Jensen and was introduced by him in order to facilitate a very fine analysis of the structure of L. The difficulty with the usual L-hierarchy is that in this hierarchy the levels are not necessarily closed under very simple set-theoretic operations, e.g., the formation of pairs. The Enumeration Theorem for Σ_1 sets depends upon pairing, however.

The idea of the Jensen hierarchy is to attain each successor level $J_{\alpha+1}$ by closing $J_\alpha \cup \{J_\alpha\}$ under a collection of basic functions, like pairing, called *rudimentary* functions. This is sufficient to capture all subsets of J_α first-order definable over J_α, and in fact these are *all* of the subsets of J_α obtained. Thus, the levels of J are very closely tied with those of L. The exact relationship is as follows: $J_0 = L_0 = \varnothing$, $J_1 = L_\omega = HF$, $L_{\omega+\alpha} = V_{\omega+\alpha} \cap J_{1+\alpha}$, $On \cap J_\alpha = \omega \cdot \alpha$, $J_\alpha = L_\alpha$ iff $\omega \cdot \alpha = \alpha$ iff ω^ω divides α.

A thorough treatment of the J-hierarchy can be found in Devlin's book [1].

1. Rudimentary functions. A function f: $V^n \to V$ is *rudimentary* if and only if it is generated by the following schemata:

(i) $f(x_1, \ldots, x_n) = x_i$, $1 \leqslant i \leqslant n$,
(ii) $f(\bar{x}) = \{x_i, x_j\}$, $1 \leqslant i, j \leqslant n$,
(iii) $f(\bar{x}) = x_i - x_j$, $1 \leqslant i, j \leqslant n$,
(iv) $f(\bar{x}) = h(g_1(\bar{x}), \ldots, g_k(\bar{x}))$,
(v) $f(y, \bar{x}) = \bigcup_{z \in y} g(z, \bar{x})$.

$R \subseteq V^n$ is *rudimentary* if and only if there is a rudimentary function f such that $\bar{x} \in R \leftrightarrow f(\bar{x}) = \varnothing$.

Examples. $x \cup y$, $\{\bar{x}\}$, $\langle x, y\rangle$, $x \in y$ are all rudimentary. If $R(y, \bar{x})$ is rudimentary, so is $f(y, \bar{x}) = y \cap \{z \mid R(z, \bar{x})\}$.

There is a nice characterization of rudimentary relations given by

Lemma 1.1. *$R \subseteq V^n$ is rudimentary if and only if R is Σ_0^{ZF}.*

(A definable relation is Σ_n^{ZF} if it has a definition provably equivalent in ZF to a formula in Σ_n form.) This fails for functions, as graph(f) rudimentary $\not\rightarrow f$ rudimentary. In fact, by 1.1, $V \times \{\omega\}$ is rudimentary, but

Lemma 1.2. *If $f(x_1, \ldots, x_n)$ is rudimentary then there exists $k < \omega$ such that* rank $f(x_1, \ldots, x_n) <$ max(rank $x_1, \ldots,$ rank $x_n) + k$ *for all* $(x_1, \ldots, x_n)$ (rank x = *least α such that $x \in V_{\alpha+1}$*).

X is *rudimentarily closed* if, for all rudimentary f: $V^n \to V$, $f''X^n \subseteq X$. For all sets X, define $\models_X^{\Sigma_n} = \{\langle i, x_1, \ldots, x_n\rangle \mid$ the ith Σ_n fmla φ is n-ary and $\langle X, \varepsilon\rangle \models \varphi(x_1, \ldots, x_n)\}$.

Lemma 1.3. *X rudimentarily closed implies $\models_X^{\Sigma_n}$ is Σ_n^X, uniformly in X.*

Lemma 1.3 is the key fact about rudimentarily closed sets and establishes the existence of a *universal* Σ_n^X set for rudimentarily closed X.

2. The J_β and S_β hierarchies. For any transitive set X, rud(X) = rud. closure $(X \cup \{X\})$ = the least rud. closed $Y \supseteq X \cup \{X\}$.

Lemma 1.4. *X transitive implies $\mathcal{P}(X) \cap$* rud(X) = Def(X), *where* Def(X) $= \{Y \subseteq X \mid Y$ *is first-order definable over* $\langle X, \varepsilon\rangle\}$.

Thus, each $z \in (\mathrm{rud}(X) - X)$ is simply obtained from first-order definable subsets of X.

Lemma 1.5. *There is a rudimentary function $\underline{S}$ such that for transitive X, $\underline{S}(X)$ is transitive, $X \cup \{X\} \subseteq \underline{S}(X)$, $\bigcup_{n\in\omega} \underline{S}^n(X) =$* rud($X$).

The S-hierarchy is defined by

$$S_0 = \varnothing, \qquad S_{\alpha+1} = \underline{S}(S_\alpha), \qquad S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha.$$

The limit levels (and only those levels) of this hierarchy are rudimentarily closed. We define

$$J_0 = \varnothing, \qquad J_{\alpha+1} = \operatorname{rud} J_\alpha, \qquad J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha.$$

So, $J_\alpha = S_{\omega\cdot\alpha}$.

LEMMA 1.6. (i) *Each* S_α, J_α *is transitive.*
(ii) $\langle S_\gamma|\gamma < \omega\cdot\alpha\rangle$ *and* $\langle J_\alpha|\gamma < \alpha\rangle$ *are uniformly* $\Sigma_1^{J_\alpha}$ *for all* α.
(iii) *On* $\cap J_\alpha = \omega\cdot\alpha$, *for all* α.

Σ_1 *well-orderings*. What makes the J_α's special among rudimentarily closed sets is the fact that they are uniformly Σ_1 well-orderable. This is the content of the following important lemma.

LEMMA 1.7. *There are well-orderings* $<_\gamma$ *of the* S_γ *such that*
(i) $\gamma_1 < \gamma_2 \to <_{\gamma_1}$ *is an initial segment of* $<_{\gamma_2}$.
(ii) $\langle <_\gamma|\gamma < \omega\cdot\alpha\rangle$ *is uniformly* $\Sigma_1^{J_\alpha}$.
(iii) *The function* $\mathrm{pr}_\alpha(x) = \{z|z <_{\omega\cdot\alpha} x\}$ *is uniformly* $\Sigma_1^{J_\alpha}$.
(iv) *Order-type* $(<_{\omega\cdot\alpha}) = \omega^\alpha$.

Note that the ordering $<_{\omega\cdot\alpha}$ of J_α has order-type ω^α, not $On \cap J_\alpha = \omega\cdot\alpha$. Thus, unlike α-recursion theory, one cannot identify members of J_α with ordinals less than $\omega\cdot\alpha$. The following lemma shows, however, that one can *nonuniformly in* α construct a Σ_1 well-ordering of J_α of length $\omega\cdot\alpha$.

LEMMA 1.8. *There is a* $\Sigma_1^{J_\alpha}$ *map of* J_α 1-1 *onto* $\omega\cdot\alpha$.

The well-ordering of Lemma 1.8 will require in general a parameter from J_α and will not satisfy property (iii) of Lemma 1.7. Property (iii) is crucial to enable the inductive construction of Σ_1 sets over J_α.

Skolem Functions. Let X be rudimentarily closed. A Σ_n *Skolem Function* for X is a Σ_n^X function f, $\operatorname{dom} f \subseteq \omega \times X$, such that if $P(y)$ is Σ_n^X with parameter p, then $\exists y P(y) \to \exists i \in \omega P(h(i, p))$.

The preceding lemmas enable one to obtain a Σ_1 Skolem Function for each J_α as follows: Define $h'(i, x) \simeq$ the least (in $<_{\omega\cdot\alpha}$) pair $\langle y, z\rangle$ such that $\psi(i, x, y, z)$, where $\exists z\psi(i, x, y, z)$ is a Σ_1 formula universal for Σ_1 formulas of two variables x, y. Then define $h(i, x) = (h'(i, x))_0$, where $(\ )_0$ is projection onto the first coordinate. Then

LEMMA 1.9. h *is a* Σ_1 *Skolem Function for* J_α.

h is called the *canonical* Σ_1 Skolem Function for J_α. Note that h is parameter-free and has a uniform definition over all J_α's.

A very deep and important result of Jensen is

THEOREM 1.10 (JENSEN'S UNIFORMIZATION THEOREM). *Each J_α has a Σ_n Skolem Function, $n \geqslant 1$ (nonuniformly).*

For $n > 2$, the Σ_n Skolem Function requires a parameter. 1.10 allows one to generalize many operations performed on Σ_1 predicates to Σ_n predicates, $n > 1$.

LEMMA 1.11. (i) (*Taking the Hull*). *Say f is a Σ_n Skolem Function for J_α with parameter p and $X \subseteq J_\alpha$ is closed under pairing. If $p_1, \ldots, p_k \in J_\alpha$, then $f''[\omega \times (X \cup \{\langle p, p_1, \ldots, p_k \rangle\})] \prec_{\Sigma_n} J_\alpha$.*

(ii) (*Inverting the Hull*). *Say $\beta < \omega \cdot \alpha$, f as in* (i). *If $p_1, \ldots, p_k \in J_\alpha$, $Y = f''[\omega \times (S_\beta \cup \{\langle p, p_1, \ldots, p_k \rangle\})]$, then there is a* 1-1 *function g: $Y \to \beta$ which is Σ_n over $\langle Y, \varepsilon \rangle$.*

In (i) above, $f''[\omega \times (X \cup \{\langle p, p_1, \ldots, p_k \rangle\}]$ is called the Σ_n *Skolem Hull* of $X \cup \{p, p_1, \ldots, p_k\}$.

LEMMA 1.12 (CONDENSATION). *If $X \prec_{\Sigma_1} J_\alpha$, then there is a unique π: $X \xrightarrow{\sim} J_\beta$ for some $\beta \leqslant \alpha$. Also, if $Y \subseteq X$ is transitive $\pi | Y = \mathrm{id} | Y$.*

Lemmas 1.11 and 1.12 are very useful in conjunction and were first exploited by Gödel in his proof of the GCH in L [**6**].

3. Cofinalities, projecta, and stability. There are important differences among the J_α's for various α, and these differences are best revealed by examining set theory inside J_α.

Let β, γ be limit ordinals, $\gamma \leqslant \beta$, $n \geqslant 1$. Then Σ_n^β-cf(γ) (Σ_n^β-*cofinality of* γ) is the least $\delta \leqslant \gamma$ such that there is an unbounded f: $\delta \to \gamma$ which is Σ_n definable over S_β. The Σ_n^β-pr(γ) (Σ_n^β-*projectum of* γ) is the least $\delta \leqslant \gamma$, such that there is a 1-1 f: $\gamma \to \delta$ which is Σ_n definable over S_β. If $\gamma = \beta$, we write Σ_n cf β, ρ_n^β for Σ_n^β-cf(β), Σ_n^β-pr(β), respectively. If $n = 1$, we write β^* for ρ_1^β.

For β, γ limit ordinals, $\gamma < \beta$, γ is a β-*cardinal* if $S_\beta \vDash$ "γ is a cardinal", γ is a *regular β-cardinal* if $S_\beta \vDash$ "γ is regular", and the *β-cardinality of* γ (*β-cofinality of* γ, resp.) equals δ if and only if $S_\beta \vDash$ "cardinality $\gamma = \delta$" ($S_\beta \vDash$ "cofinality $\gamma = \delta$", resp.).

PROPOSITION 1.13. (i) *Σ_n^β-pr(γ) is a β-cardinal for $\gamma < \beta$. $\rho_n^\beta = \beta$ or is a β-cardinal.*

(ii) *Σ_n^β-cf(γ) is a regular β-cardinal for $\gamma < \beta$. $\Sigma_n$cf $\beta = \beta$ or is a regular β-cardinal.*

β is Σ_n-*admissible* if and only if $\Sigma_n$cf $\beta = \beta$ and Σ_n-*nonprojectible* if and only if $\rho_n^\beta = \beta$. An important characterization of ρ_n^β is given by the following result.

THEOREM 1.14. *ρ_n^β = the least δ such that there is a set $A \subseteq \delta$ which is Σ_n-definable over S_β but not a member of S_β.*

PROOF. We use 1.10, 1.11, and 1.12. Let f be a Σ_n Skolem Function with parameter p, $\delta < \beta$. Suppose there is a $A \subseteq \delta$, $A \notin S_\beta$, and A is Σ_n over S_β with parameter q. Let $X = f''[\omega \times (S_\delta \cup \{\langle p, q\rangle\})] \prec_{\Sigma_n} S_\beta$. By 1.12, let π: $X \simeq S_\gamma$, $\gamma < \beta$. $A \subseteq \delta$ is definable over X, hence over S_γ, since $\pi|\delta = \mathrm{id}|\delta$. Since $A \notin S_\beta$, we must have $\gamma = \beta$. But then by 1.11(ii), $\rho_n^\beta \leqslant \delta$.

It remains to show there exists $A \subseteq \rho_n^\beta$, $A \notin S_\beta$ and A is Σ_n over S_β. By 1.8, let g: $S_\beta \to \beta$ be 1-1 and onto, g Σ_1 over S_β. Let h: $\beta \to \rho_n^\beta$ be 1-1, Σ_n over S_β. Then define $A \subseteq \rho_n^\beta$ by $\delta \in A \leftrightarrow \exists x[h \circ g(x) = \delta$ and $\delta \notin x]$. Then A is Σ_n over S_β. If $A \in S_\beta$, then if $\delta_0 = h \circ g(A)$, we have

$$\delta_0 \in A \leftrightarrow \delta_0 \notin (h \circ g)^{-1}(\delta_0) \leftrightarrow \delta_0 \notin A. \quad \dashv$$

We now state some results which are special to the Σ_1 case and arise from the fact that Σ_0, unlike Σ_n for $n > 0$, is closed under complementation.

LEMMA 1.15. *γ a β-cardinal, $\gamma > \omega$ implies $S_\gamma \prec_{\Sigma_1} S_\beta$ (i.e., γ is β-stable).*

PROOF. Suppose $\delta < \gamma$, $\phi(x)$ is Δ_0 with parameters from S_δ, and $S_\beta \vDash \exists x \phi(x)$. We want to show $S_\gamma \vDash \exists x \phi(x)$.

Case 1. β is a limit of limit ordinals.

Choose a limit ordinal $\beta' < \beta$ such that $S_{\beta'} \vDash \exists x\phi(x)$. Let h be the canonical Σ_1 Skolem Function for $S_{\beta'}$. Let $X = h''(\omega \times S_\delta)$. By 1.12, let π: $X \simeq S_\alpha$. Since $\pi|\delta = \mathrm{id}|\delta$, $S_\alpha \vDash \exists x\phi(x)$. But by 1.11(ii), β-card $(\alpha) \leqslant \delta < \gamma$. So $\alpha < \gamma$ and $S_\gamma \vDash \exists x \phi(x)$.

Case 2. $\beta = \beta' + \omega$.

Suppose $(\exists x \in S_{\beta'+n})\phi(x)$. Let $\exists z \phi(i, x, y, z)$ be a Σ_1 formula universal for Σ_1 formulas of the variables, x, y. Define the partial function h, $\operatorname{dom} h \subseteq \omega \times S_\delta$, by $h(i, x) \simeq y$ if $\exists z[\langle y, z\rangle \in S_{\beta'+n}$ and $\langle y, z\rangle$ is the $<_\beta$-least pair such that $\phi(i, x, y, z)]$. Let X = range h. It is easy to show that $Y = X \cap S_{\beta'} \prec_{\Sigma_1} S_{\beta'}$. Let π: $Y \simeq S_\alpha$, by 1.12. As in Case 1, $\alpha < \gamma$. Since $X \subseteq \mathrm{rud}(S_{\beta'})$, there exists $Z \subseteq S_{\alpha+n}$, Z transitive, such that $\langle X, \varepsilon\rangle \simeq \langle Z, \varepsilon\rangle$. But then $\langle Z, \varepsilon\rangle \vDash \exists x\phi(x)$, and so $S_{\alpha+n} \vDash \exists x \phi(x)$. Since $\alpha + n < \gamma$, $S_\gamma \vDash \exists x\phi(x)$. $\dashv$

PROPOSITION 1.16. *If β is not (Σ_1)-admissible, then*

(i) $S_\beta \vDash \exists$ largest cardinal.

(ii) $\beta^* < \beta$.

PROOF. Let f: $\gamma_0 \to \beta$ be order-preserving, unbounded, $\gamma_0 < \beta$, and f be Σ_1 over S_β with parameter p.

(i) If γ is a β-cardinal such that $\gamma_0, p \in S_\gamma$, then by 1.15, range $f \subseteq \gamma$, contradiction.

(ii) Let γ = largest β-cardinal. For $\alpha < \gamma_0$, let g_α be the $<_\beta$-least injection of $S_{f(\alpha+1)} - S_{f(\alpha)}$ into γ. Define $f(x) = \langle y, z\rangle$ if and only if (y = least y' such that $x \in S_{f(y'+1)} - S_{f(y')}$ and $g_y(x) = z$). Then f is a Σ_1^β injection of S_β into $\gamma_0 \times \gamma$. Let $g \in S_\beta$ map $\gamma_0 \times \gamma$ 1-1 into γ. Then $g \circ f$ injects β into $\gamma < \beta$. $\dashv$

Note that in the proof of 1.16(ii), we could have actually shown that there is a Σ_1^β map of β 1-1 *onto* γ = greatest β-cardinal. Define $\hat{\beta}$ = least δ such that there is a Σ_1^β bijection of β onto δ. Thus,

PROPOSITION 1.17. *β is admissible if and only if $\hat{\beta} = \beta$.*

There is no analogous result for Σ_2-admissibility, as $\aleph_\omega^L$ is not Σ_2-admissible, but certainly not Σ_2-projectible.

Every successor β-cardinal is β-regular; the proof is the usual one in set theory with the axiom of choice. However, if a successor β-cardinal is $\leqslant \beta^*$, we get more:

PROPOSITION 1.18. *Suppose $\lambda \leqslant \beta^*$ is a successor β-cardinal. Then λ is Σ_1^β-regular, i.e., λ is regular with respect to functions Σ_1 over S_β.*

PROOF. If $\lambda < \beta^*$, then any Σ_1^β function with domain and range $\subseteq \lambda$ is a member of S_β by 1.14. So the proposition is clear in this case.

Suppose $f\colon \gamma \xrightarrow{\text{o.p}} \beta^*$, $\gamma < \beta^*$, and $f(x) = y$ is defined by $\exists z\phi(x, y, z)$ over S_β. Also, assume $\mu^+ = \beta^*$ (so that β^* is a successor β-cardinal) and for each $\alpha < \beta^*$, let g_α = the $<_\beta$-least in injection of α into μ. Now if f is unbounded, then define $g\colon \beta^* \to \gamma \times \mu$ by

$$g(x) = \langle \gamma_1, \delta \rangle \text{ iff } \quad \exists z_1\, \exists z_2\, \exists \alpha_1\, \exists \alpha_2 [\phi(\gamma_1, \alpha_1, z_1) \wedge \phi(\gamma_1 + 1, \alpha_2, z_2) \\ \wedge \alpha_1 \leqslant x < \alpha_2 \wedge g_{\alpha_2}(x) = \delta].$$

Then g is Σ_1^β and $g\colon \beta^* \xrightarrow{\text{1-1}} \gamma \times \mu$, contradicting the definition of β^*. $\dashv$

Note. It *may* happen that β^* is a regular β-cardinal but β^* is *not* regular with respect to functions Σ_1 over S_β. For example, let κ be strongly inaccessible in L and define $\beta_0 = \kappa$, $\beta_n = \kappa + \omega \cdot n$. Also let $\kappa_0 = \omega$, $\kappa_{n+1} = \sup[(\Sigma_1\text{- Skolem Hull } (\omega \times (\kappa_n \cup \{\kappa\})) \text{ in } S_{\beta_{n+1}}) \cap \kappa]$. Then $\kappa_\omega + \omega^2$ is such a β where $\kappa_\omega = \bigcup_n \kappa_n$. In this case $\beta^* = \kappa_\omega$ is a regular β-cardinal but the sequence $\kappa_0, \kappa_1, \kappa_2, \ldots$ is β-recursive. This example is due to Fred Abramson.

CHAPTER 2. THE KEY DEFINITIONS

Let β be a limit ordinal. We define β-recursion theory to be the study of sets Σ_1 definable over S_β. The following definitions coincide with those from α-recursion theory (recursion theory on Σ_1 admissible ordinals):

$A \subseteq S_\beta$ is *β-recursive enumerable* if A is Σ_1 definable over S_β.

$A \subseteq S_\beta$ is *β-recursive* if both A and $S_\beta - A$ are β-recursively enumerable.

We abbreviate β-recursively enumerable as β-r.e. and β-recursive as β-rec. Recursively enumerable and recursive sets were defined initially for sets of natural numbers as those sets enumerable or computable by an algorithm; i.e., a procedure given by a finite machine. The recursively enumerable sets of integers coincide with those Σ_1-Definable over HF = the hereditarily finite sets. The above definition is a generalization of this definability-theoretic viewpoint of recursively enumerable set. As in α-recursion theory the notion of finite undergoes alteration:

$A \subseteq S_\beta$ is *β-finite* if $A \in S_\beta$.

A function $f\colon A \to S_\beta$, $A \subseteq S_\beta$ is β-rec. if graph(f) is β-rec., and f is β-finite if graph(f) is β-finite.

At this point, a few remarks are in order concerning our choice of definitions. β-recursion theory certainly should capture the idea of performing "computations" of length $< \beta$. Accordingly, certain basic relations and functions such as $< \restriction \beta \times \beta$ and ordinal successor should be taken to be β-computable, as well as relations obtained from these by closure under bounded quantification $\forall x < \gamma$, for $\gamma < \beta$. From this one can show that all relations Δ_0 over $\langle S_\beta, \varepsilon\rangle$ must be β-computable.

Our notion of β-recursively enumerable should capture the idea of being listable in a β-computable way. As any set Σ_1 over $\langle S_\beta, \varepsilon\rangle$ is the range of a function whose graph is Δ_0 over $\langle S_\beta, \varepsilon\rangle$, we see that all β-r.e. sets in fact are β-computably listable. But in practice any set which has a β-computable listing turns out to be Σ_1-definable over $\langle S_\beta, \varepsilon\rangle$.

Our notion of β-finite can similarly be justified in this way as the β-finite sets are exactly those β-r.e. sets which can be listed in fewer than β steps. The situation with β-recursiveness is less clear. There are certainly reasonable and stronger conditions for a set to be β-recursive. For example, for $A \subseteq S_\beta$, define $A^* = \{z \subseteq A \mid z \text{ is } \beta\text{-finite}\}$. We may require that not only A, $S_\beta - A$ be β-r.e. but also A^*, $(S_\beta - A)^*$ be β-r.e. in order for A to be β-recursive. However, there is an a posteriori justification for our definition in that it is the β-recursive sets as we have defined them that play a key role in the classification of the β-r.e. sets. So making further restrictions on β-recursiveness would only necessitate inventing a new term for those sets which are both β-r.e. and co-β-r.e.

Enumerations. In α-recursion theory, an enumeration of an α-r.e. set A is just an α-rec. function whose range is A. It follows from the Σ_1-admissibility of α that one can effectively compute each initial segment of such an enumeration, uniformly. In β-recursion theory, the notion of enumeration requires greater scrutiny.

An *enumeration* of a β-r.e. set $A \subseteq S_\beta$ is a β-rec. function $f\colon S_\beta \to A$ such that Range $f = A$ and for some limited formula $\phi(x, y, z)$, we have, for sufficiently large $\delta < \beta$,

$$f(x) = y \leftrightarrow \exists z \in S_\delta \phi(x, y, z)$$

for all $x, y \in S_\delta$. Thus $f|S_\delta$ is uniformly Σ_1 over S_δ, for δ sufficiently large.

Any nonempty β-r.e. set has an enumeration, for if A is defined by $x \in A \leftrightarrow \exists y \phi(x, y)$, ϕ limited, let

$$f(z) = \begin{cases} x, & \text{if } z = \langle x, y \rangle \text{ and } \phi(x, y), \\ \text{least member of } A, & \text{otherwise.} \end{cases}$$

Then f is an enumeration of A.

It will be convenient to discuss enumerations in terms of their increasing sequence of ranges $\{\text{Range } f|S_\delta\}$. Thus by an "enumeration" $\{E_\delta\}$, we mean such a sequence arising from an enumeration f.

An enumeration $\{E_\delta\}$ is *terminating* if for some δ_0, $E_\delta = E_{\delta_0}$, for all $\delta \geqslant \delta_0$. Clearly, the β-finite sets are exactly the β-r.e. sets which have a terminating enumeration. But *not* every enumeration of a β-finite set need terminate, and this is an important difference between β-recursion theory and α-recursion theory. For example, when constructing β-r.e. sets via an enumeration, there may be β-finite subsets of the resulting β-r.e. set whose enumeration is never completed at any stage of the construction. In the next chapter it is shown that for many β, this phenomenon must be present when constructing nontrivial β-r.e. sets.

Another principle from α-recursion theory which fails in this context is: "An α-recursive subset of an α-finite set is α-finite." At first sight, this appears to be an inadmissible pathology, but further thought reveals its plausibility: Although for each member of the given β-finite set one can effectively decide if it belongs to the given β-recursive subset, one cannot necessarily make all of those decisions by some bounded stage. This occurs for every inadmissible β as a consequence of the existence of a β-recursive 1-1 correspondence of β with some smaller ordinal.

Reducibilities. As S_β is rudimentarily closed, we may obtain a β-recursive enumeration $\{\phi_e(x)\}_{e \in S_\beta}$ of the Σ_1 formulas with parameters from S_β and sole free variable x. (We identify formulas with their gödel numbers.) Then

$$\phi(e, x) \leftrightarrow \exists \delta < \beta \langle S_\delta, \varepsilon \rangle \vDash \phi_e(x)$$

is a Σ_1 formula universal for Σ_1 formulas of one free variable. We let $W_e = \{x | S_\beta \vDash \phi_e(x)\}$, the eth β-r.e. set. Clearly $C = \{\langle e, x \rangle | S_\beta \vDash \phi(e, x)\}$ is a complete β-r.e. set.

Let $\{e\}$ be the partial Σ_1 function defined by

$$\{e\}(x) \simeq y \leftrightarrow \exists \delta [\langle y, \delta \rangle = <_\beta\text{-least pair s.t. } \langle S_\delta, \varepsilon \rangle \vDash \phi_e(\langle x, y \rangle)].$$

Also, for $A \subseteq S_\beta$,

$$\{e\}^A(x) \simeq y \leftrightarrow \exists z_1 \exists z_2 [z_1, z_2 \in S_\beta, z_1 \subseteq A, z_2 \subseteq S_\beta - A \text{ and}$$
$$\langle z_1, z_2, y\rangle = <_\beta\text{-least triple s.t. } \{e\}(\langle z_1, z_2, x\rangle) \simeq y]. \quad (*)$$

Clearly $\{e\}^A$ is single-valued (though not necessarily total). A function f is *weakly β-reducible to A* ($f \leqslant_{w\beta} A$) iff $f \simeq \{e\}^A$ for some e. $B \subseteq S_\beta$ is *weakly β-reducible to A* ($B \leqslant_{w\beta} A$) if $C_B = \{e\}^A$ for some e, where C_B = the characteristic function of B. Also, B is *β-reducible to A* ($B \leqslant_\beta A$) iff there exists an e such that

$$\left.\begin{array}{l} z_1 \subseteq B \leftrightarrow \{e\}^A(z_1) \simeq 0, \\ z_2 \subseteq S_\beta - B \leftrightarrow \{e\}^A(z_2) \simeq 1, \end{array}\right. \qquad z_1, z_2 \in S_\beta.$$

Lastly, B is *finitely β-reducible to A* ($B \leqslant_{f\beta} A$) iff $\exists e\ C_B \simeq \{e\}_f^A$ where $\{e\}_f^A$ is defined as in (*) above, except where z_1, z_2 range over *finite* sets.

A pair $\langle z_1, z_2\rangle$ where $z_1 \subseteq A$, $z_2 \subseteq S_\beta - A$ is termed a *membership fact* about A. A membership fact $\langle z_1, z_2\rangle$ is finite if and only if z_1 and z_2 are finite. Thus $B \leqslant_{w\beta} A$ if all finite membership facts about B can be effectively generated from (β-finite) membership facts about A, $B \leqslant_\beta A$ if all membership facts about B can be effectively generated from membership facts about A, and $B \leqslant_{f\beta} A$ if finite membership facts about B can be effectively generated from finite membership facts about A.

$\leqslant_{w\alpha}$ was introduced originally as the reducibility when α is admissible, but due to its asymmetric definition was later discovered to be intransitive for $\alpha = \omega_1^{CK}$ = first nonrecursive ordinal.

$\leqslant_\beta$ is transitive and has evolved to be the correct reducibility for β-recursion theory. If α is admissible, many theorems about Turing degrees have been lifted to $\leqslant_\alpha$-degrees. A word of warning: $A \leqslant_\beta \varnothing$ is *not* the same as A β-recursive. Thus there may be different β-degrees of β-recursive sets. (In the next chapter we will show that this occurs exactly when β is not admissible.)

$\leqslant_{f\beta}$ is introduced primarily as a technical device in various proofs. It is transitive but does not capture the full flavor of β-recursion theory as all membership facts are finite.

Tamely-r.e., strongly-r.e. and regular sets. We wish to study the β-degrees of β-r.e. sets, where for $A \subseteq S_\beta$, the β-degree of $A = \{B \mid B \leqslant_\beta A$ and $A \leqslant_\beta B\}$. In case β is admissible, this study has gone quite far with the use of priority method, invented in ordinary recursion theory by Friedberg [2] and Muchnik [13] to prove the existence of incomparable Turing degrees of r.e. sets of integers. In a priority construction, the desired properties of the r.e. sets we wish to build are expressed in terms of a collection of requirements.

These requirements are then arranged into a well-ordered list, a requirement lower in the list having "higher priority" than those above it. In the case of Post's Problem for an admissible α, where α-r.e. sets A, B of incomparable α-degree are constructed, a typical requirement is satisfied by placing some x into B (A, respectively) and establishing a computation $\{e\}^A(x) = 0$ ($\{e\}^B(x) = 0$, respectively); accordingly, certain α-finite sets z_1, z_2 are sought so that $z_1 \subseteq A$, $z_2 \subseteq L_\alpha - A$ ($z_1 \subseteq B$, $z_2 \subseteq L_\alpha - B$, respectively). Conflicts arise between requirements because, for example, we may wish to place some x' into A for some requirement R', but requirement R may require $z_2 \subseteq L_\alpha - A$ where $x' \in z_2$. The priority listing gives preference to requirements of higher priority and one then shows inductively that each requirement is permanently satisfied beyond some stage of the construction.

When $\alpha > \omega$, the above method encounters difficulties as the final induction is Σ_2 over L_α and not Σ_1 over L_α. Sacks and Simpson **[17]** employ an argument using Σ_1-elementary substructures to perform this induction for arbitrary Σ_1-admissibles. If one would like to adapt this approach to the case of an inadmissible β, then the problems with induction are even greater, but there is another added difficulty: Computations $\{e\}^A(x) = 0$ depend on β-finite sets $z_1 \subseteq A$, $z_2 \subseteq S_\beta - A$. In the admissible case, there will be some stage σ in the construction where, if A^σ = part of A enumerated by stage σ, we will have $z_1 \subseteq A^\sigma$, $z_2 \subseteq S_\beta - A \subseteq S_\beta - A^\sigma$ and thus $\{e\}^{A^\sigma}(x) = 0$. This can fail in the inadmissible case, where $z_1 \subseteq A$ does *not* imply $\exists\sigma(z_1 \subseteq A^\sigma)$. So the computation $\{e\}^A(x) = 1$ may never be apparent at any stage of the construction.

Thus we would like our enumeration $\{A^\sigma\}$ of A to have the following property: If $K \subseteq A$ is β-finite, then $\exists\sigma(K \subseteq A^\sigma)$. This property of $\{A^\sigma\}$ is called *tameness*. Now if we arrange our construction so that the resulting enumeration of A is tame, then all computations from A will appear at some stage in the construction.

We define

A is *tamely-r.e.* (t.r.e.) iff A has an enumeration $\{A^\sigma\}$ s.t. $K \subseteq A$, K β-finite $\to K \subseteq A^\sigma$ for some σ.

Of course, even if A is t.r.e., there may be (and will be if β is inadmissible) enumerations of A which are not tame. If β is admissible, any enumeration is tame and thus t.r.e. = r.e.

PROPOSITION 2.1. *A is t.r.e. $\leftrightarrow A^* = \{K | K \subseteq A\}$ is r.e. $\leftrightarrow A^*$ is t.r.e.*

PROOF. If A is t.r.e., let $\{A^\sigma\}$ be a tame enumeration of A. Then $K \subseteq A \leftrightarrow \exists\sigma(K \subseteq A^\sigma)$ so A^* is r.e. Also, $K \subseteq A^* \leftrightarrow \bigcup K \subseteq A \leftrightarrow \exists\sigma(\bigcup K \subseteq A^\sigma)$ so A^* is t.r.e.

Suppose A^* is r.e. We show that A is t.r.e. Let $\{A^*_\sigma\}$ be an enumeration of

A^*. If $A^\sigma = \bigcup A^*_\sigma$, then $\{A^\sigma\}$ is an enumeration of A. But $K \subseteq A \to K \in A^* \to \exists\sigma(K \in A^*_\sigma) \to \exists\sigma(K \subseteq A^\sigma)$, so $\{A^\sigma\}$ is tame. $\dashv$

The statement "$\{K|K \subseteq A\}$ is r.e." is equivalent to the statement that the collection of certain limited sentences (i.e., no unbounded quantifiers) which are true about A is r.e. Let $\Delta_0(A)$ be the set of all sentences with parameters from S_β which consist of a string of bounded quantifiers followed by a quantifier-free matrix in which A occurs as a predicate. Also, let $\Delta_0^+(A) \subseteq \Delta_0(A)$ consist of those sentences in $\Delta_0(A)$ in which all occurrences of A are *positive*, i.e., $x \in A$ is within the scope of an even number of negation signs whenever it occurs. Then $\varphi \in \Delta_0^+(A)$, $\varphi(A)$ true, $A \subseteq B$ implies $\varphi(B)$ true. Also, if $\varphi \in \Delta_0(A)$, $\delta < \beta$ and all parameters in φ belong to S_δ, then $\varphi(A) \leftrightarrow \varphi(A \cap S_\delta)$.

Now, A is t.r.e. iff $\{\varphi \in \Delta_0^+(A)|\varphi(A)$ is true and φ is of the form $\forall x \in K(x \in A)\}$ is r.e. If β is admissible, then $\{\varphi \in \Delta_0^+(A)|\varphi$ is true$\}$ is an r.e. set (we identify sentences with their Gödel numbers $\in S_\beta$). We define

A is *strongly-r.e.* (s.r.e.) iff the true sentences of $\Delta_0^+(A)$ form an r.e. set. Also, let A be

n-r.e. iff the true sentences of $\Delta_0^+(A)$ involving $(n-1)$ alternations of bounded quantifiers form an r.e. set, for $n > 0$. So s.r.e. $\leftrightarrow \forall n$ (n-r.e.), and 1-r.e.$\to$ t.r.e. If β is admissible, s.r.e.$=$ r.e.

$A \subseteq S_\beta$ is *regular* if $z \in S_\beta$ implies $A \cap z \in S_\beta$. Regularity is important as it allows one to get a universal Σ_1 predicate for $\langle S_\beta, \varepsilon, A\rangle$, and thus do recursion theory relative to A. A theorem of Sacks **[16]** in α-recursion theory says that any α-r.e. set has the same α-degree as some regular α-r.e. set. We shall prove a regular sets theorem for s.r.e. sets.

PROPOSITION 2.2. *If A is t.r.e. and regular, then A is s.r.e.*

PROOF. If $\phi(A) \in \Delta_0^+(A)$, then

$$\begin{aligned}\phi(A) \text{ is true} &\leftrightarrow \exists\delta < \beta[\phi(A \cap S_\delta) \text{ is true}]\\ &\leftrightarrow \exists z[\phi(z) \text{ is true and } z \subseteq A].\end{aligned}$$

Since A is t.r.e, this last equivalence is an r.e. predicate of ϕ. $\dashv$

CHAPTER 3. DEGREES OF T.R.E. SETS

1. A regular sets theorem. The regular sets theorem of Sacks **[16]** states that for admissible α, every α-r.e. set has the same α-degree as some regular α-r.e. set. Admissibility is used heavily, and in fact, we shall see that this theorem is false for some inadmissible β. However, a version is true for s.r.e. sets:

THEOREM 3.1. *Suppose β is admissible or $\beta^* = \mathrm{gc}\,\beta$, the largest β-cardinal. Then every s.r.e. set has the same β-degree as a regular s.r.e. set.*[3]

[3]We thank W. Maass for pointing out the necessity of the assumption "$\beta^* = \mathrm{gc}\,\beta$" in our proof of 3.1. Maass has proved 3.1 without this assumption. See **[12]**.

The regular sets theorem is useful in α-recursion theory, as regular α-r.e. sets are easier to work with than nonregular ones. Results later in this chapter indicate that nonregularity is an essential feature of β-recursion theory.

PROOF OF 3.1. The proof is similar to the one in α-recursion theory (see Simpson [**20**]).

LEMMA 3.2. *Let A be s.r.e. Then there is a set A^*, $A^* \equiv_\beta A$, A^* s.r.e. such that for all C, $A^* \leqslant_\beta C$ iff $A^* \leqslant_{w\beta} C$.*

PROOF OF 3.2. Let $A^* = \{z | A \cap z \neq \varnothing\}$. A^* is s.r.e. since any sentence in $\Delta_0^+(A^*)$ is effectively equivalent to a sentence in $\Delta_0^+(A)$.

$A \leqslant_\beta A^*$ because:

(i) $z \subseteq S_\beta - A \leftrightarrow z \notin A^*$.

(ii) $z \subseteq A$ is an r.e. (even s.r.e.) predicate.

$A^* \leqslant_\beta A$ because:

(i) $z \subseteq S_\beta - A^* \leftrightarrow \bigcup z \subseteq S_\beta - A$.

(ii) $z \subseteq A^*$ is an r.e. (even s.r.e.) predicate.

Lastly, suppose $A^* \leqslant_{w\beta} C$. As before, $z \subseteq A^*$ is s.r.e., so we need only concern ourselves with the negative part of the reduction $A^* \leqslant_\beta C$. But

$$z \subseteq S_\beta - A^* \leftrightarrow \bigcup z \notin A^*$$

and since $A^* \leqslant_{w\beta} C$, we can determine the r.h.s. from β-finite membership facts on C. ⊣

We assume that the given s.r.e. set A has the property of A^* in the lemma.

We can assume β is inadmissible (by Sacks' Theorem), so by 1.16, there is a largest β-cardinal. By assumption, it is β^*.

Let $\gamma_0 = \mu\gamma(A \cap S_\gamma$ is not β-finite). Let k be a β-finite bijection from S_{γ_0} onto β^*. Since A is t.r.e., $k[A \cap S_{\gamma_0}]$ has a tame enumeration f': $S_\beta \to k[A \cap S_{\gamma_0}] = A_1$.

We define a 1-1 tame enumeration f: $S_\beta \to \beta^* \times \beta^*$ as follows: Let $z_0, z_1, \ldots, z_\alpha, \ldots$ be the members of S_β in $<_\beta$-increasing order such that $f'(z_\alpha) \notin f'[\{z | z <_\beta z_\alpha\}]$. Let f_α be the $<_\beta$-least injection of $\{z | z_\alpha <_\beta z <_\beta z_{\alpha+1}\}$ into $\beta^* - \{0\}$ (since $\beta^* = \mathrm{gc}\,\beta$). Define

$$f(z) = \langle f'(z_\alpha), f_\alpha(z)\rangle \quad \text{if } z_\alpha <_\beta z <_\beta z_{\alpha+1},$$
$$f(z_\alpha) = \langle f'(z_\alpha), 0\rangle.$$

Thus f simply "fills out" f' so as to make it 1-1. Order $\beta^* \times \beta^*$ as follows: $(\alpha, \gamma) < (\alpha', \gamma')$ iff $\alpha < \alpha'$ or ($\alpha = \alpha'$ and $\gamma < \gamma'$).

We can assume that A has a 1-1 tame enumeration (if not, replace A by $\{\langle 0, x\rangle | x \in S_\beta\} \cup \{\langle 1, x\rangle | x \in A\}$); let h be one. Then $g = f \circ h$ is a 1-1 tame enumeration of $B = f[A] \subseteq A_1 \times \beta^*$.

The *deficiency set* of g is $D_g = \{z | \exists z' >_\beta z\ g(z') < g(z)\}$. We show that D_g is the desired set in 9 steps:

(a) D_g is regular.

CLAIM. For any z, $\exists z' >_\beta z$ s.t. $\forall z'' >_\beta z'$,

$$\{w <_\beta z | g(z') < g(w)\} = \{w <_\beta z | g(z'') < g(w)\}.$$

PROOF OF CLAIM. Otherwise, there is a sequence $g(z') > g(z'') > g(z''') > \ldots.$ $\dashv$

Given z, choose z' as in claim. Then

$$\{w <_\beta z | w \in D_g\} = \{w <_\beta z | g(z') < g(w)\}$$
$$\cup \{w <_\beta z | \exists w', w <_\beta w' <_\beta z', \text{ s.t. } g(w') < g(w)\}.$$

Both sets on the right are β-finite.

(b) D_g is t.r.e.

$$z \subseteq D_g \leftrightarrow \exists z'[z \subseteq \{w <_\beta z' | g(z') < g(w)\}$$
$$\cup \{w <_\beta z' | \exists w', w <_\beta w' <_\beta z', g(w') < g(w)\}].$$

Thus, by 2.2, D_g is s.r.e.

(c) $A \leqslant_{f\beta} B.\ z \in A \leftrightarrow f(z) \in B$.

(d) $A_1 \cap \alpha$, $B \cap \alpha$ are β-finite for all $\alpha < \beta^*$. This is clear since A_1, B are r.e.

(e) $g[S_\beta - D_g]$ is unbounded in $\beta^* \times \beta^*$.

Let $\alpha < \beta^*$. We will find $w \in S_\beta - D_g$ such that $g(w) \geqslant (\alpha, 0)$.

Since g is tame and (by (d)) $B \cap \alpha$ is β-finite, we can choose x_1 so that for all $x \geqslant_\beta x_1$, $g(x) > \alpha$. Let $\alpha' = <$ -least member of $g[\{y | x_1 \leqslant_\beta y\}]$. Then let $w = <_\beta$-least $w' \geqslant_\beta x_1$ such that $g(w') = \alpha'$. Then $w \in S_\beta - D_g$ and $g(w) > (\alpha, 0)$. $\dashv$

(f) $B \leqslant_{f\beta} D_g$. $z \notin B \leftrightarrow z \notin \beta^* \times \beta^*$ or $\exists w \in S_\beta - D_g[g(w) > z$ and $\forall w' <_\beta w\ g(w') \neq z]$.

(g) $A \leqslant_\beta D_g$. By (c) and (f), $A \leqslant_{w\beta} D_g$. By choice of A, $A \leqslant_{w\beta} D_g \rightarrow A \leqslant_\beta D_g$.

(h) $z \subseteq S_\beta - D_g$, z β-finite $\rightarrow g[z]$ is bounded in $\beta^* \times \beta^*$.

g is order-preserving on $S_\beta - D_g$, $S_\beta - D_g$ is unbounded, and $g[S_\beta - D_g]$ is unbounded in $\beta^* \times \beta^*$.

(i) $D_g \leqslant_\beta A$. $z \subseteq S_\beta - D_g \leftrightarrow \bigcup_{z' \in z}[\{(\alpha_1, \alpha_2) | (\alpha_1, \alpha_2) < g(z')\} - g[\{w | w <_\beta z'\}]] \subseteq \beta^* \times \beta^* - B$. Denote this union by z_0. Then z_0 is bounded in $\beta^* \times \beta^*$ by (h). By (d), $z_0 \cap A_1$ is β-finite, so $z \subseteq S_\beta - D_g \leftrightarrow f^{-1}[z_0 \cap A_1] \subseteq S_\beta - A$. Since $A_1 \leqslant_\beta A$, we are done. $\dashv$

The s.r.e.-ness of the original set is needed to apply 3.2. But full s.r.e.-ness is not needed, but only 2-r.e.-ness. So

COROLLARY 3.3. *If A is 2-r.e. then there exists a regular s.r.e. set B such that $A \equiv_\beta B$ (under the assumptions of 3.1).*

Questions. Is there a nice characterization of the β-degrees of regular β-r.e. sets? For which β does the regular sets theorem hold for all β-r.e. sets?

We will later exhibit a β where there are β-r.e. sets *not* of the same β-degree as any regular set.

2. Upper bounds for the t.r.e. degrees. It is not apparent that there is an abundance of t.r.e. sets, or even a nonrecursive t.r.e. set when β is inadmissible. Of course, for admissible α, t.r.e = r.e. and the results of α-recursion theory show that the structure of the t.r.e. degrees is quite rich. In this section, we show that for some inadmissible β's, the t.r.e. sets are all β-reducible to $\varnothing$ and that for every inadmissible β there is an incomplete upper bound to all the t.r.e. degrees. Thus in general, tameness is a very strong assumption.

THEOREM 3.4. *Assume β is not admissible. Then there is a β-recursive set A such that*

(i) $0 <_\beta A <_\beta C$ *where* $C =$ *complete β-r.e. set*;

(ii) *any t.r.e. set and any β-recursive set is β-reducible to A*;

(iii) $C \leqslant_{\mathrm{w}\beta} A$.

PROOF. Let $\gamma = \Sigma_1 \mathrm{cf}\, \beta$. Let $f\colon \gamma \to \beta$ be β-recursive, Range f unbounded. Define $A = \{\langle e, x, \alpha\rangle | \{e\}(x)$ converges by stage $f(\alpha)\} = \{\langle e, x, \alpha\rangle | S_{f(\alpha)} \models \phi_e(x)$ is defined$\}$. A is clearly Δ_1 over S_β, hence β-recursive.

LEMMA 3.5. $B \leqslant_{\mathrm{w}\beta} A \to B$ *is* Δ_2 *over* S_β.

PROOF. Say

$$z \in B \leftrightarrow \exists z_1\, \exists z_2 [\phi(z, z_1, z_2) \simeq 0 \wedge z_1 \subseteq A \wedge z_2 \subseteq S_\beta - A]$$

where ϕ is Σ_1. The predicate on the right is Σ_2 since

$$z_1 \subseteq A \leftrightarrow \forall w \in z_1 (w \in A) \in \Pi_1,$$

$$z_2 \subseteq S_\beta - A \leftrightarrow \forall w \in z_2 (w \in S_\beta - A) \in \Pi_1.$$

Similarly for $S_\beta - B$. $\dashv$

LEMMA 3.6. *Let* $C_2 =$ *complete* Σ_2 *set for* S_β, $C =$ *complete* Σ_1 *set for* S_β. *Then* $C_2 \leqslant_{\mathrm{w}\beta} C$.

PROOF OF 3.6. Say $z \in C_2 \leftrightarrow \exists z_1 \forall z_2 R(z, z_1, z_2)$ with R a limited formula. Define $\phi(z, z_1) \simeq z_2$ if and only if $z_2 = <_\beta$-least w such that $\sim R(z, z_1, w)$. Let $f(z) =$ index for $\lambda z_1 \phi(z, z_1)$ as a partial Σ_1 function, f β-recursive. Then

$$z \notin C_2 \leftrightarrow \forall z_1\, \exists z_2 \sim R(z, z_1, z_2) \leftrightarrow \{f(z)\} \text{ total}.$$

Let $\hat{\beta}$ be as defined after Proposition 1.16. By 1.8, there is a β-recursive, 1-1, onto $g\colon \hat{\beta} \to \beta$. Let h be β-recursive such that $\{h(e)\} \simeq \{e\} \circ g$. Then $z \notin C_2 \leftrightarrow \{h(f(z))\}$ is total on $\hat{\beta} \leftrightarrow \{h(f(z))\} \times \hat{\beta} \subseteq C = \{\langle e, x\rangle | \{e\}(x)$ is defined$\}$. Also, if e_0 is such that $\langle e_0, x\rangle \in C$ if and only if $\sim \forall z_2 R(x_0, x_1, z_2)$

(where $x = \langle x_0, x_1 \rangle$), then

$$z \in C_2 \leftrightarrow \exists z_1[\forall z_2 R(z, z_1, z_2)] \leftrightarrow \exists z_1[\langle e_0, \langle z, z_1 \rangle\rangle \notin C].$$

So $C_2 \leqslant_{w\beta} C$. ⊣

LEMMA 3.7. $C \leqslant_{w\beta} A$.

PROOF OF 3.7. $C = \{\langle e, x \rangle | \{e\}(x) \text{ is defined}\}$. $\langle e, x \rangle \notin C \leftrightarrow \{e\} \times \{x\} \times \gamma \subseteq S_\beta - A$. ⊣

PROOF OF (i). $0 <_\beta A$, because otherwise, by 3.7, $C \leqslant_{w\beta} A \leqslant_\beta 0 \rightarrow C \leqslant_{w\beta} 0 \rightarrow C$ is Δ_1, contradiction.

$A <_\beta C$, because otherwise, by 3.6, $C_2 \leqslant_{w\beta} C \leqslant_\beta A \rightarrow C_2 \leqslant_{w\beta} A$, contradicting 3.5. ⊣

PROOF OF (ii). Say B is t.r.e. Then $z \subseteq B$ is Σ_1. Also $z \subseteq S_\beta - B \leftrightarrow \{e_0\} \times z \times \gamma \subseteq S_\beta - A$, for some e_0. So $B \leqslant_\beta A$.

Say B is β-recursive. Then

$$z \subseteq B \leftrightarrow \{e_1\} \times z \times \gamma \subseteq S_\beta - A,$$
$$z \subseteq S_\beta - B \leftrightarrow \{e_2\} \times z \times \gamma \subseteq S_\beta - A,$$

where e_1, e_2 are indices for $S_\beta - B$, B, respectively as Σ_1 sets. ⊣

PROPOSITION 3.8. *Suppose $\Sigma_1\text{cf}\,\beta < \beta^*$ and $S_\beta \models$ "β^* = greatest cardinal is a successor cardinal." Then every t.r.e set is regular.*

PROOF. Say A is t.r.e. If A is not regular, then there exists δ such that $A \cap S_\delta$ is a t.r.e. non-β-finite subset of S_δ. Since $\beta^* = \text{gc}\,\beta$, there is a β-finite f: $S_\delta \leftrightarrow \beta^*$, 1-1 onto. Then $B = f[A \cap S_\delta]$ is a t.r.e. non-β-finite subset of β^*. Let g: $S_\beta \rightarrow B$ be a tame enumeration of B.

Let h: $\Sigma_1\text{cf}\,\beta \rightarrow S_\beta$ be such that $\forall\alpha\; g(h(\alpha)) \notin g[\{w | w <_\beta h(\alpha)\}]$, Range h is unbounded in $<_\beta$, and h is β-recursive. h exists since otherwise B must be β-finite. By 1.18, $C = \text{Range}\; g \circ h$ is bounded in β^* and hence is a β-finite subset of B. But then g is not tame since $\forall\alpha$, $C \not\subseteq g[\{w | w <_\beta h(\alpha)\}]$. ⊣

Note. The hypothesis $\beta^* = \text{gc}\,\beta$ can be eliminated from 3.8.

PROPOSITION 3.9. *Suppose $\Sigma_1\text{cf}\,\beta < \beta^*$. Then every regular t.r.e. set is recursive in $\varnothing$.*

PROOF (MAASS, AFTER R. SHORE [19]). Let $\gamma = \Sigma_1\text{cf}\,\beta$, h: $\gamma \rightarrow \beta$, Range h unbounded, h β-recursive. Let A be regular, t.r.e., and $\{A_\delta\}$ be a tame enumeration of A. Define $P \subseteq \gamma \times \gamma$ by:

$$P(\alpha_1, \alpha_2) \leftrightarrow \forall\delta \geqslant h(\alpha_2)[h(\alpha_1) \cap (A_\delta - A_{h(\alpha_2)}) = \varnothing].$$

Then $P(\alpha_1, \alpha_2) \leftrightarrow [A \cap h(\alpha_1) = A_{h(\alpha_2)} \cap h(\alpha_1)]$. Since P is a Π_1 over S_β subset of $\gamma \times \gamma$, and $\gamma < \beta^*$, by 1.14, P is β-finite. But then

$$K \subseteq A \leftrightarrow \exists \alpha_1 \exists \alpha_2 \big[K \subseteq h(\alpha_1) \cap A_{h(\alpha_2)} \text{ and } P(\alpha_1, \alpha_2) \big],$$
$$K \subseteq S_\beta - A \leftrightarrow \exists \alpha_1 \exists \alpha_2 \big[K \subseteq h(\alpha_1) - A_{h(\alpha_2)} \text{ and } P(\alpha_1, \alpha_2) \big],$$

so $A \leqslant_\beta \varnothing$. ⊣

COROLLARY 3.10. *Suppose $\beta = \aleph_1^L + \omega$. Then every t.r.e. set is β-reducible to $\varnothing$.*

3. Post's Problem when $\Sigma_1\mathrm{cf}\,\beta \geqslant \beta^*$. We use the method of blocking, invented by R. Shore [18], to solve Post's Problem when $\Sigma_1\mathrm{cf}\,\beta \geqslant \beta^*$.

Post's Problem. Show that there are β-r.e. sets A, B such that $A \not\leqslant_{w\beta} B$, $B \not\leqslant_{w\beta} A$.

The original solution to Post's Problem in α-recursion theory (Sacks and Simpson [17]) is not as easily adapted to this context as the later blocking proof found in [21]. The reason is that the following lemma (Lemma 2.3 of [17]) may fail:

LEMMA. *Let α be admissible, κ an infinite regular α-cardinal, $\gamma < \kappa$ and $\{A_\rho | \rho < \gamma\}$ an α-r.e. sequence of α-r.e. sets. If* α-card. $A_\rho < \kappa$ *for all $\rho < \gamma$ (so in particular each A_ρ is α-finite) then $\bigcup_\rho A_\rho$ is α-finite and has α-cardinality $< \kappa$.*

PROPOSITION 3.11. *There is a limit ordinal β, Σ_1*cf *$\beta \geqslant \beta^* > \omega$, and a β-r.e. sequence $\{A_n | n < \omega\}$ such that* β-card. $A_n = \omega$ *for all n, but $\bigcup_{n<\omega} A_n = \beta$.*

PROOF. Define inductively:
$H_0 = \Sigma_1$-Skolem Hull of $\{\aleph_1^L\}$ inside $L_{\aleph_1^L + \aleph_1^L}$,
$\alpha_1 = H_0 \cap \aleph_1^L =$ an ordinal,
$H_{n+1} = \Sigma_1$-Skolem Hull of $\{\alpha_{n+1}, \aleph_1^L\}$ inside $L_{\aleph_1^L + \aleph_1^L}$,
$\alpha_{n+2} = H_{n+1} \cap \aleph_1^L =$ an ordinal.
Let $\alpha = \sup \alpha_n$, $\beta = \alpha + \alpha$. $L_\beta =$ the transitive collapse of $\bigcup_n H_n \prec_{\Sigma_1} L_{\aleph_1^L + \aleph_1^L}$. Under this collapse, α corresponds to $\aleph_1^L$.

CLAIM. $\beta^* = \alpha = \aleph_1^{L_\beta}$.

PROOF. Otherwise, there exists $f\colon \alpha \xrightarrow{1\text{-}1} \omega$, f is Σ_1 over L_β. Let π: $H = \bigcup_n H_n \xrightarrow{\simeq} L_\beta$, and if $p \in L_\beta$ is the parameter defining f, let $q = \pi^{-1}(p)$. Then $q \in H_n \to \mathrm{Dom}\, f \subseteq H_n \to \alpha \subseteq \alpha_n$, contradiction. ⊣

Since $\Sigma_1\mathrm{cf}\,\beta = \Sigma_1\mathrm{cf}\,\alpha$, by 1.18 we have $\Sigma_1\mathrm{cf}\,\beta = \alpha = \aleph_1^{L_\beta}$.

Define $A_n = \pi[H_n]$, for all n.

CLAIM. $\{A_n | n \in \omega\}$ is β-r.e.

PROOF. For each $\sigma < \beta$, define inductively
$A_0^\sigma = \Sigma_1$-Skolem Hull of $\{\aleph_1^L\}$ inside L_σ,
$\alpha_1^\sigma = A_0^\sigma \cap \alpha =$ an ordinal $< \alpha$.
$A_{n+1}^\sigma = \Sigma_1$-Skolem Hull of $\{\aleph_1^L, \alpha_{n+1}^\sigma\}$ inside L_σ,
$\alpha_{n+1}^\sigma = A_{n+1}^\sigma \cap \alpha =$ an ordinal $< \alpha$.

Now, $\bigcup_\sigma A_0^\sigma = A_0$. But since β-card. $A_0 = \omega$, $\exists\sigma\ A_0^\sigma = A_0$. Thus $\exists\sigma$ $\alpha_1^\sigma = \alpha_1$. Thus $\exists\sigma\ A_1^\sigma = A_1$. Inductively, $\forall n\ \exists\sigma\ A_n^\sigma = A_n$. Thus $\{A_n | n \in \omega\}$ is β-r.e. $\dashv$

Note. Despite this counterexample, the lemma holds when $\Sigma_1\mathrm{cf}\ \beta > \kappa$.

DEFINITION. Let $\gamma \leqslant \beta$. A function f: $\gamma \to \beta$ is tame Δ_2 if f is the limit of a "tamely convergent" β-recursive sequence of β-recursive functions, i.e., there is a β-recursive $g(\alpha, \sigma)$: $\gamma \times \beta \to \beta$ such that

(i) For all $\alpha < \gamma$, there is a "stage" σ such that, for $\alpha' \leqslant \alpha$ and any $\sigma' \geqslant \sigma$, $g(\alpha', \sigma') = g(\alpha', \sigma)$.

(ii) $f = \lim g$, i.e., $\forall\alpha\ \exists\sigma\ \forall\sigma' \geqslant \sigma[f(\alpha) = g(\alpha, \sigma')]$.

For $\delta \leqslant \beta$, the T-$\Delta_2$cf δ (tame Δ_2 cofinality of δ) is the least γ such that there is a tame Δ_2 g: $\gamma \to \delta$, $\bigcup$ Range $g = \delta$. Tame Δ_2 functions were introduced by Lerman (*On suborderings of the α-r.e. α-degrees*, Ann. Math. Logic **4** (1972)) to give a unified treatment of Post's Problem for admissible ordinals.

LEMMA 3.12. *T-$\Delta_2$cf $\beta^* = T$-$\Delta_2$cf β, if $\Sigma_1$cf $\beta \geqslant \beta^*$.*

PROOF. Let f: $\beta \to \beta^*$ be 1-1, β-recursive, and h: $\alpha_0 = T$-$\Delta_2$cf $\beta^* \to \beta^*$ be unbounded, order-preserving, and T-Δ_2. Define h': $\alpha_0 \to \beta$ by

$$h'(\alpha) = \mu\gamma\big[f[\gamma] \cap h(\alpha) = \mathrm{Range}\, f \cap h(\alpha)\big].$$

h' can be approximated by $h'(\alpha, \sigma) = \mu\gamma < \sigma[f[\gamma] \cap h(\sigma, \alpha) = f[\sigma] \cap h(\alpha)]$ where $h(\sigma, \alpha)$ is a tame approximation to h. But the approximation $h'(\sigma, \alpha)$ is tame, since (by the assumption $\Sigma_1$cf $\beta \geqslant \beta^*$) $\forall\alpha\ \exists\sigma[f[\sigma] \cap h(\alpha) = \mathrm{Range}\, f \cap h(\alpha)]$.

But h' is unbounded since f is total and 1-1. Thus T-$\Delta_2$cf $\beta = T - \Delta_2$ cf(α_0) $= \alpha_0$. $\dashv$

From now on in this section assume $\Sigma_1$cf $\beta \geqslant \beta^*$. We now describe how Shore blocking may be used to solve Post's Problem.

We would like to construct β-r.e. sets A, B such that $A \nleqslant_{\mathrm{w}\beta} B$, $B \nleqslant_{\mathrm{w}\beta} A$. Thus we would like to satisfy requirements.

$$\left.\begin{array}{l} R_e^A\colon \{e\}^A \neq B, \\ \\ R_e^B\colon \{e\}^B \neq A, \end{array}\right. \qquad \text{for each } e \in S_\beta.$$

Let

$$B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B. \end{cases}$$

An attempt at R_e^A consists of setting up a negative requirement $z \subseteq S_\beta - A$ and putting some x into B in order to insure $B(x) = 1 \neq \{e\}^A(x)$ (z is the negative part of A used in the computation $\{e\}^A(x) \simeq 0$). Attempts at R_e^B are defined similarly.

There are conflicts between these attempts, as we may want some $y \notin A$ to insure a computation $\{e\}^A(w) \simeq 0$ for some w, yet we may wish to put y into A in order to insure $\{e'\}^B(y) = 0 \neq A(y) = 1$. However, there are *no* conflicts between $R^A_{e_1}$ and $R^A_{e_2}$, as our attempts for them are to keep elements out of A and put elements into B. Thus if our priority listing of requirements is arranged into blocks $\mathfrak{B}^A_1, \mathfrak{B}^B_2, \mathfrak{B}^A_3, \mathfrak{B}^B_4, \ldots$ with requirements of type R^A_e (R^B_e) in blocks $\mathfrak{B}^A_{2\alpha+1}$ ($\mathfrak{B}^B_{2\alpha}$) then there will be no conflicts within a block.

Of course, our blocks must be ordered in some list–how long should the list of blocks be? We would like it as short as possible, so we use the T-Δ_2 cofinality of β^*. That is, let H: $\alpha_0 = T\text{-}\Delta_2\text{cf}\ \beta^* \to \beta^*$ be T-Δ_2 and order-preserving. Then the requirements are arranged in a list of length β^*, but such that requirements of rank γ where $H(2\alpha) \leqslant \gamma < H(2\alpha + 1)$ are of type R^B_e and where $H(2\alpha + 1) \leqslant \gamma < H(2\alpha + 2)$ are of type R^A_e.

Define $g(\delta) =$ "sup of the stages at which requirements of rank $< \delta$ are acted upon." Let $\gamma = \mu\delta[g(\delta) = \beta]$. We would like $\gamma = \beta^*$. Suppose $H(\alpha) < \gamma \leqslant H(\alpha + 1)$ for some α. But then we have a contradiction for there are no conflicts between requirements of rank in $[H(\alpha), \gamma)$, and since $\Sigma_1\text{cf}\ \beta > \gamma$, there will be a stage at which all of these requirements will have been acted upon. Thus $\gamma = H(\lambda)$ for some limit λ. But $g \circ H$ is T-Δ_2 on λ, so it cannot be unbounded since $\lambda < \alpha_0 = T\text{-}\Delta_2\text{cf}\ \beta^* = T\text{-}\Delta_2\text{cf}\ \beta$.

So, we arrange our requirement as above. Of course, at each stage we will only have the correct priority listing on some proper initial segment, but each proper initial segment will eventually be correctly ordered.

There is one last feature of the construction which needs comment. We must add requirements to insure that A and B are t.r.e. They are

$$T^A_e\colon e \subseteq A \to \exists\sigma\ e \subseteq A^\sigma, \qquad T^B_e\colon e \subseteq B \to \exists\sigma\ e \subseteq B^\sigma.$$

Here, A^σ (B^σ) = amount of A (B) enumerated by stage σ. An attempt is made at T^B_e by setting up the negative requirement of keeping some member of e out of A (similarly for B). Thus requirements T^A_e (T^B_e) are blocked with R^A_e (R^B_e).

The construction. The construction of A and B takes place in "stages" σ; $A^{<\sigma}$ ($B^{<\sigma}$) = amount of A (B) enumerated before stage σ, A^σ (B^σ) = $A^{<(\sigma+1)}$ ($B^{<(\sigma+1)}$) where $\sigma + 1$ = next stage after σ. Unlike α-recursion theory, we may *not* identify stages in S_β with ordinals in S_β. By a *stage* we mean a position in the well-ordering $<_\beta$, which has length ω^γ if $\beta = \omega \cdot \gamma$. However, we think of stages as ordinals $< \omega^\gamma$. Of course, the sets A and B will be Σ_1 over S_β: replace ordinal stages by the members of S_β of the corresponding ranks in $<_\beta$.

Let H: $\alpha_0 = T\text{-}\Delta_2\text{cf}\ \beta^* \to \beta^*$ be as above and $H(\sigma, \alpha)$ be a tame approximation to H. We assume that $\omega^{H(\sigma,\alpha)} = H(\sigma, \alpha)$ for all σ, α, $H(\sigma, 0) = 0$ for all σ, and $H(\sigma_0, \alpha)$ in an order-preserving function of α for fixed σ_0.

Let L: $S_\beta \to \beta^*$ be an enumeration such that $L^{-1}(\{\delta\})$ is unbounded in $<_\beta$, for all $\delta < \beta^*$. An example of such an L is

$$L(z) = \begin{cases} \delta & \text{if } z = \langle \delta, y \rangle \text{ for some } y, \delta < \beta^*, \\ 0 & \text{otherwise.} \end{cases}$$

L is used to determine which requirement to examine at a given stage.

Also let f: $S_\beta \overset{1\text{-}1}{\to} \beta^*$ be β-recursive. Let $\{f^\delta\}_{\delta<\beta}$ be an enumeration of Graph(f).

If e is a reduction procedure, recall that $\{e\}(x) \simeq y \leftrightarrow \exists\delta[\langle\delta, y\rangle = <_\beta$-least pair such that $\langle S_\delta, \varepsilon\rangle \vDash \phi_e(\langle x, y\rangle)]$. Write $\phi_e(x) = \exists z \psi_e(x, z)$. Then define

$$\{e\}_\sigma(x) \simeq y \leftrightarrow \{e\}(x) \simeq y \text{ and } \exists z <_\beta \sigma(\psi_e(\langle x, y\rangle, z)).$$

Then $\{e\}_\sigma^{A^{<\sigma}}(x) \simeq y$ means $\exists z_1 \exists z_2[z_1 \subseteq A^{<\sigma}, z_2 \subseteq S_\beta - A^{<\sigma}$ and $\{e\}_\sigma(\langle z_1, z_2, x\rangle) \simeq y]$. $A^{<\sigma}$ may change, so $\{e\}_\sigma^{A^{<\sigma}}(x) \simeq y \not\to \{e\}^A(x) \simeq y$. But if A is t.r.e., $\{e\}^A(x) \simeq y \to \exists\sigma\{e\}_\sigma^{A^{<\sigma}}(x) \simeq y$.

We also will need the auxiliary function $b(\sigma, \alpha)$: $S_\beta \times \alpha_0 \to S_\beta$ which gives a bound (in $<_\beta$) on elements mentioned by stage σ in positive or negative requirements for the sake of requirements of rank $< H(\sigma, \alpha)$. b is used to preserve negative requirements.

Lastly, we introduce the Witness Function Proviso (WFP) after Sacks and Simpson [17]. This says that witnesses for attempts to satisfy requirements $R_{e_1}^{A[B]}$, $R_{e_2}^{A[B]}$, $e_1 \neq e_2$, should come from disjoint sets, so that an x which is a candidate for $\{e_1\}^A(x) \neq B(x)$ ($\{e_1\}^B(x) \neq A(x)$) should never be a candidate for $\{e_2\}^A(x) \neq B(x)$ ($\{e_2\}^B(x) \neq A(x)$) when $e_1 \neq e_2$. So if L': $S_\beta \to \beta^*$ is a fixed enumeration such that $(L')^{-1}(\{\delta\})$ is unbounded in $<_\beta$ $\forall\delta < \beta^*$, then WFP requires that all x's put into A or B for the sake of an attempt to satisfy the requirements of rank δ should belong to $(L')^{-1}(\{\delta\})$. This insures that at each stage, each requirement R_e^A has a plentiful supply of arguments x on which to attempt $\{e\}^A(x) \neq B(x)$.

We are now ready to describe the construction.

Set $b(0, \alpha) = 0$ for all α. $A^0 = B^0 = \varnothing$.

Stage σ. Let $L(\sigma) = \delta$. Choose $\alpha < \alpha_0$ such that $H(\sigma, \alpha) \leqslant \delta < H(\sigma, \alpha + 1)$. We would like requirements in each block $[H(\alpha), H(\alpha + 1))$ to examine all reduction procedures e such that $f(e) < H(\alpha + 1)$.

So, let c be the $<_\beta$-least map of $[H(\sigma, \alpha), H(\sigma, \alpha + 1))$ 1-1 onto $H(\sigma, \alpha + 1)$. c exists since we can assume $\omega^{H(\sigma,\alpha+1)} = H(\sigma, \alpha + 1)$.

Let $f^\sigma = f^\gamma \restriction \{w | w \leqslant_\beta \sigma\}$ where $\gamma = \mu\gamma'[\sigma \in S_{\gamma'}]$. Then f^σ = part of Graph(f) enumerated by stage σ. If $c(\delta) \notin$ Range f^σ, go to the next stage.

If $f^\sigma(e) = c(\delta)$, we consider the reduction procedure e.

α *is even*. We make attempts at R_e^B and T_e^B. Let $\langle x, z_1, z_2\rangle = <_\beta$-least $\langle x, z_1, z_2\rangle$ such that $z_1 \subseteq B^{<\sigma}$, $z_2 \subseteq S_\beta - B^{<\sigma}$, $\{e\}_\sigma(\langle z_1, z_2, x\rangle) \simeq 0$ and

$x \geqslant b(\sigma, \alpha)$, $L'(x) = c(\delta)$(WFP). This says that $\{e\}^{B^{<\sigma}}(x) \simeq 0$ via a neighborhood condition (z_1, z_2) and x does not interfere with WFP or any requirement of rank $< H(\sigma, \alpha)$. Let $y = <_\beta$-least y such that $y \in e - B^{<\sigma}$. If there is an $x' \in A^{<\sigma}$, $\{e\}_\sigma^{B^{<\sigma}}(x') \simeq 0$, $x' < b(\sigma, \alpha)$, $L'(x') = c(\delta)$, then let

$$A^\sigma = A^{<\sigma},$$

$$b(\sigma + 1, \alpha') = \begin{cases} b(\sigma, \alpha'), & \alpha' \leqslant \alpha, \\ <_\beta\text{-maximum}(y, b(\sigma, \alpha')) + 1, & \alpha' > \alpha. \end{cases}$$

Otherwise, let

$$A^\sigma = A^{<\sigma} \cup \{x\},$$

$$b(\sigma + 1, \alpha') = \begin{cases} b(\sigma, a'), & \alpha' \leqslant \alpha, \\ <_\beta\text{-supremum}(z_2 \cup \{y\} \cup \{b(\sigma, \alpha')\}) + 1, & \alpha' > \alpha. \end{cases}$$

Note. $(\sigma + 1) = <_\beta$-immediate successor to σ. The definition of b insures that no member of $z_2 \cup \{y\}$ will be put into B by a requirement of higher rank (lower priority).

α is odd. Identical to α is even, except switch A and B.

It is understood that if no such $\langle x, z_1, z_2 \rangle$ and no such y exist in the above, then one simply goes to the next stage and $b(\sigma, \text{-})$ does not change, $A^\sigma = A^{<\sigma}$, $B^\sigma = B^{<\sigma}$.

This ends the construction.

CLAIM 1. $\forall \alpha < \alpha_0$, $\lim_\sigma b(\sigma, \alpha)$ exists.

PROOF. Let $\alpha = \mu\alpha$ ($\lim_\sigma b(\sigma, \alpha)$ does not exist). We argue toward a contradiction.

$\alpha = 0$. Since $H(\sigma, 0) = 0\ \forall\sigma$, we have $b(\sigma, 0) = 0\ \forall\sigma$.

$\alpha = \alpha' + 1$. Pick a stage σ_0 such that $\forall\sigma \geqslant_\beta \sigma_0$.

(i) $H(\sigma, \alpha'') = H(\alpha'')$ for $\alpha'' \leqslant \alpha$.

(ii) $b(\sigma, \alpha') = \lim_\sigma b(\sigma, \alpha')$.

σ_0 exists since H is tame and, by choice of α, $\lim_\sigma b(\sigma, \alpha')$ exists.

Now $b(\sigma, \alpha)$ can change at most twice for each $\delta \in [H(\alpha'), H(\alpha' + 1))$ (once for R_e^B and once for T_e^B if $f(e) = \delta$ and α is even; replace B by A if α is odd). Since $\Sigma_1\text{cf}\ \beta > H(\alpha' + 1)$, there is a stage σ_1 beyond which $b(\sigma, \alpha)$ cannot change.

α a limit ordinal. Let $b'\colon \alpha \to \beta$ be defined by

$$b'(\alpha) = \mu\sigma\left[b(\sigma', \alpha') = \lim_\sigma b(\sigma, \alpha')\ \forall\sigma' \geqslant_\beta \sigma\right].$$

Then b' is T-Δ_2, and since $\alpha < \alpha_0 = T\text{-}\Delta_2\text{cf}\ \beta^* = T\text{-}\Delta_2\text{cf}\ \beta$, b' is bounded. But then $b'(\alpha) = \sup_{\alpha' < \alpha} b'(\alpha)$ exists and so $\lim_\sigma b(\sigma, \alpha)$ exists. $\dashv$

Let $b(\alpha) = \lim_\sigma b(\sigma, \alpha)$, $\alpha < \alpha_0$.

CLAIM 2. A and B are t.r.e.

PROOF. We show that T_e^A and T_e^B are satisfied. Let $f(e) = \delta$. Choose α such that $H(\alpha) \leqslant \delta < H(\alpha + 1)$ and let σ_0 be such that $\forall\sigma \geqslant_\beta \sigma_0$,

(i) $H(\sigma, \alpha') = H(\alpha') \, \forall \alpha' \leqslant \alpha + 2$,
(ii) $b(\sigma, \alpha') = b(\alpha') \, \forall \alpha' \leqslant \alpha + 2$.

Assume α is even, without loss of generality. Let $c_1 = <_\beta$-least map of $[H(\alpha), H(\alpha + 1))$ 1-1 onto $H(\alpha + 1)$, and $c_2 = <_\beta$-least map of $[H(\alpha + 1), H(\alpha + 2))$ 1-1 onto $H(\alpha + 2)$. Let $\delta_1 = c_1^{-1}(\delta)$, $\delta_2 = c_2^{-1}(\delta)$.

If $e \not\subseteq B^\sigma \forall \sigma$ ($e \not\subseteq A^\sigma \forall \sigma$), then for some $\sigma \geqslant_\beta \sigma_0$, $L(\sigma) = \delta_1$ (δ_2), an attempt is made at T_e^B (T_e^A). Since $b(\sigma, \alpha) = b(\alpha)$ $(b(\sigma, \alpha + 1) = b(\alpha + 1))$, any such attempt will be permanent, so $e \not\subseteq B$ $(e \not\subseteq A)$.

So A, B are t.r.e. $\dashv$

CLAIM 3. $A \not\leqslant_{w\beta} B$, $B \not\leqslant_{w\beta} A$.

PROOF. Pick a reduction procedure e and let $f(e) = \delta$. Choose α even, $H(\alpha + 1) > \delta$ and let $c_1 = <_\beta$-least map of $[H(\alpha), H(\alpha + 1))$ 1-1 onto $H(\alpha + 1)$. Let $\delta_1 = c_1^{-1}(\delta)$.

Since $\lim_\sigma b(\sigma, \alpha + 1)$ exists and $\{x | L'(x) = \delta\}$ is unbounded in $<_\beta$, the WFP implies that $Y = \{x | L'(x) = \delta \text{ and } x \notin A\}$ is unbounded in $<_\beta$. If $\{e\}^B = A$, pick $x_0 \in Y$, $x_0 > b(\alpha + 1)$ and a stage σ, $L(\sigma) = \delta_1$, such that

(i) $\{e\}_\sigma^{B^{<\sigma}}(x_0) \simeq A(x_0) = 0$.
(ii) $\forall \sigma' \geqslant_\beta \sigma \; b(\sigma', \alpha + 1) = b(\alpha + 1)$, $b(\sigma', \alpha) = b(\alpha)$.

But then at stage σ, the least such x_0 would have been put into A (contradicting (ii)) unless there is an $x' \in A^{<\sigma}$, $\{e\}^{B^{<\sigma}}(x') \simeq 0$, $x' < b(\sigma, \alpha)$. But by (ii) again, $\{e\}_\sigma^{B^{<\sigma}}(x') \simeq \{e\}^B(x') \not\simeq A(x') = 1$. This contradiction shows $\{e\}^B \neq A$. Similarly $\{e\}^A \neq B$. $\dashv$

CLAIM 4. A and B are regular.

PROOF. Let $z \in S_\beta$. Choose $\alpha < \alpha_0$ such that $b(\alpha) >_\beta z$ (this is possible because there are requirements $T_{\{w\}}^A$ which cause Range b to be unbounded in $<_\beta$). Choose a stage σ_0 such that $\forall \sigma \geqslant_\beta \sigma_0$, $b(\sigma, \alpha) = b(\alpha)$ and $H(\sigma, \alpha) = H(\alpha)$. Then no requirement of rank $\geqslant H(\alpha + 1)$ can put an element $w \leqslant_\beta z$ into A or B after stage σ_0. Thus $A \cap \{w | w <_\beta z\}$ and $B \cap \{w | w <_\beta z\}$ are β-finite. $\dashv$

By 2.2, we have proved

THEOREM 3.13. *If* $\Sigma_1$cf $\beta \geqslant \beta^*$, *then there exist regular s.r.e. sets* A, B *such that* $A \not\leqslant_{w\beta} B$, $B \not\leqslant_{w\beta} A$.

CHAPTER 4. SIMPLE β-R.E. SETS

$A \subseteq \beta^*$ is *simple* if $\beta^* - A$ has order type β^* and for every β-r.e. set $B \subseteq \beta^*$, B unbounded in $\beta^* \to A \cap B \neq \varnothing$. Making a set simple is the easiest way to make it nonrecursive and simplicity is often used for this reason. When $\beta^* = \beta$, every β-r.e. set has the same β-degree as some simple β-r.e. set–take the deficiency set of some enumeration (see **[21]**).

The construction of simple α-r.e. sets when α is admissible is fairly easy. A similar construction works for arbitrary β, but to show this, we shall need a

β-recursive analogue of

FODOR'S THEOREM. *If κ is an uncountable regular cardinal and f: $A \to \kappa$, $A \subseteq \kappa$ then $\{\alpha < \kappa | f[A \cap \alpha] \subseteq \alpha\}$ is a closed unbounded subset of κ.*

Our β-recursive analogue holds for arbitrary β-recursively regular cardinals, but we shall only need

THEOREM 4.1. *If $\kappa \leqslant \beta^*$ is a successor β-cardinal and f is a partial β-recursive function*, Dom f, Range $f \subseteq \kappa$, *then $\{\alpha < \kappa | f[\text{Dom} f \cap \alpha] \subseteq \alpha\}$ is a closed unbounded subset of κ.*

DEFINITION. $A \subseteq \kappa$ is *closed* if $\lambda < \kappa$, $A \cap \lambda$ unbounded in λ implies $\lambda \in A$.

PROOF OF 4.1. Let $\kappa = \lambda^+$. Choose a limit ordinal $\alpha_0 > \lambda$, $\alpha_0 < \kappa$. Let p be the parameter needed to define f as a partial β-recursive function and let $H = \Sigma_1$-Skolem Hull $(\alpha_0 \cup \{p\})$ in S_β.

CLAIM 1. $H \cap \kappa =$ an ordinal (call it α).

PROOF. If $\gamma < \kappa$, $\gamma \in H$, then $\exists f \in H(f\colon \gamma \overset{1-1}{\to} \lambda)$. Since $\lambda \subseteq H$, Dom $f =$ Range $f^{-1} = \gamma \subseteq H$.

CLAIM 2. $\alpha < \kappa$.

PROOF. By 1.11(ii), there is a function f: $H \overset{1\text{-}1}{\to} \alpha_0$ which is Σ_1 over H, hence over S_β. Thus it suffices to show that there is no β-recursive g: $\kappa \overset{1\text{-}1}{\to} \lambda$ (since β-cardinality $(\alpha_0) = \lambda$). But since $\lambda < \kappa \leqslant \beta^*$ the range of any such g is β-finite by 1.14. Let $h = g^{-1}$. Then h shows that κ is not regular with respect to β-recursive functions, contradicting 1.18.

CLAIM 3. $f[\text{Dom} f \cap \alpha] \subseteq \alpha$.

PROOF. Since $H \prec_{\Sigma_1} S_\beta$, certainly $f[\text{Dom} f \cap \alpha] \subseteq H$. But $H \cap k = \alpha$. $\dashv$

The simple set construction.[4] For each $e \in S_\beta$ we have the requirement: S_e: $W_e \subseteq \beta^*$, W_e unbounded in β^* implies $A \cap W_e \neq \varnothing$. These requirements are easy enough to meet; conflicts arise because in addition we would like to have order type $(\beta^* - A) = \beta^*$. As before, we use a β-recursive f: $\beta \overset{1\text{-}1}{\to} \beta^*$ to list our requirements S_e.

Let L: $S_\beta \to \beta^*$ be an enumeration such that $L^{-1}(\{\delta\})$ is unbounded in $<_\beta$ for all $\delta < \beta^*$.

Stage σ. $A^{<\sigma} =$ part of A enumerated so far. If $L(\sigma) \notin$ Range f^σ, go to the next stage. Otherwise, let $f^\sigma(e) = L(\sigma)$. If $W_e^\sigma \cap A^{<\sigma} \neq \varnothing$, go to the next stage.

Otherwise, define $x = \mu y[y \in W_e^\sigma$ and $y > L(\sigma) + L(\sigma)]$, if such a y exists. Then let $A^\sigma = A^{<\sigma} \cup \{x\}$. This completes the construction.

[4]The technique used here can be used to simplify Simpson's generalization of Dekker's Theorem (every nonzero r.e. degree contains a simple r.e. set). Simpson's result appears as Lemma 2.9, p. 59 in [20].

CLAIM 1. For all e, S_e is satisfied.

PROOF. Say $f(e) = \delta$. If $W_e \subseteq \beta^*$ is unbounded in β^*, then at some stage σ, $f^\sigma(e) = \delta$, $L(\sigma) = \delta$ and W_e^σ has a member $> \delta + \delta$. But then at stage σ, either $A^{<\sigma} \cap W_e^\sigma \neq \varnothing$ or some $x \in W_e$ was put into A.

CLAIM 2. $\beta^* - A$ has order type β^*.

PROOF. *Case* 1. $\beta^* > \omega$. It suffices to show: κ a successor β-cardinal, $\kappa \leqslant \beta^*$ implies $\kappa - A$ has o.t. κ.

Note that for $\delta < \kappa$, there is at most *one* x put into A for the sake of S_e, where $f(e) = \delta$. Define

$g(\delta) \simeq x$ iff $\exists e[f(e) = \delta$ and x is put into A for the sake of $S_e]$. Then by 4.1, $X = \{\alpha < \kappa \mid g[\mathrm{Dom}\, g \cap \alpha] \subseteq \alpha\}$ is closed unbounded in κ. But $\alpha \in X$ implies $[\alpha, \alpha + \alpha] \subseteq \kappa - A$, so o.t. $(\kappa - A) = \kappa$. $\dashv$

Case 2. $\beta^* = \omega$. In this case we want $\beta^* - A$ infinite. Fix $n < \omega$. There is a $k \leqslant n + n$, $k \geqslant n$ such that k is never put into A for the sake of S_e, $f(e) < n$ (by a counting argument). But then $k \notin A$. So A is infinite. $\dashv$

T.R.E. subsets of β^*.

THEOREM 4.2. *There exists a t.r.e. subset of* β^* *which is not* β*-finite* $\leftrightarrow T\text{-}\Delta_2\mathrm{cf}\,\beta = T\text{-}\Delta_2\mathrm{cf}\,\beta^*$.

PROOF OF ($\rightarrow$). Say $A \subseteq \beta^*$, A t.r.e. but not β-finite. Let $\delta_1 = T\text{-}\Delta_2\mathrm{cf}\,\beta^*$ and let f_1 be a $T\text{-}\Delta_2$ function mapping δ_1 order-preservingly onto an unbounded subset of β^*.

Let f be a tame enumeration of A.

Define g: $\delta_1 \rightarrow \beta$ as follows:

$$g(\delta) = \mu\sigma[f[\sigma] \cap f_1(\delta) = A \cap f_1(\delta)].$$

g is well defined since f is tame and is actually $T\text{-}\Delta_2$ since f_1 is. If g were bounded, then $A = f[\sigma]$ for some σ, contradicting A not being β-finite. Lastly, g is order-preserving. Thus $T\text{-}\Delta_2\mathrm{cf}\,\beta = T\text{-}\Delta_2\mathrm{cf}\,\delta_1 = \delta_1$. $\dashv$

PROOF OF ($\leftarrow$). Here we have two types of requirements. The first ones are T_e: $e \subseteq A \rightarrow \exists\sigma(e \subseteq A^\sigma)$. These are just the tameness requirements from the proof of 3.13. In order to insure $A \subseteq \beta^*$ is not β-finite, we use simplicity requirements as above, except here we only require the weaker S_e: $e \subseteq \beta^*$, o.t. $e = \beta^* \rightarrow A \cap e \neq \varnothing$. The important difference is that we deal only with β-finite sets and not β-r.e. sets. Of course, this suffices to show that A is not β-finite.

An attempt is made at T_e as follows: If $e \nsubseteq A^{<\sigma}$, pick δ = least member of $e - A^{<\sigma}$ and set up the negative requirement $\delta \notin A$.

Attempts at S_e are as before: If $e \subseteq \beta^*$, o.t. $e = \beta^*$, then put into A the least member of e exceeding $\delta + \delta$, where δ = height of S_e in the listing of requirements.

To show that the requirements T_e will be met, we will need to know that the activity of requirements of higher priority eventually ceases (as in admissibility theory). We can achieve this by using a very nice projection of S_β into β^* to define the priority listing. Since the S_e requirements deal only with β-finite sets, we can bound the activity of a proper initial segment of the priority listing as soon as we bound the reduction procedures e which are represented by this proper initial segment.

LEMMA 4.3. *Say T-$\Delta_2\mathrm{cf}\,\beta = T$-$\Delta_2\mathrm{cf}\,\beta^*$. Then there is a β-recursive p': $S_\beta \times S_\beta \to \beta^*$ such that*

(i) *p' is convergent, i.e., for some p: $S_\beta \to \beta^*$, $\forall\alpha \lim_\sigma p'(\sigma, \alpha) = p(\alpha)$.*

(ii) *$p = \lim p'$ is 1-1.*

(iii) *$\forall\delta < \beta^*\ \exists\sigma[p^{-1}[\delta] \subseteq \sigma$ and $\forall\sigma' \geqslant_\beta \sigma\ \forall\alpha(p(\alpha) < \delta \leftrightarrow p'(\sigma', \alpha) < \delta)]$.*

PROOF. Choose a β-finite c: $\beta^* \times \beta^* \xrightarrow{1\text{-}1} \beta^*$, such that for all $\delta_1, \delta_2 < \beta^*$, $c(\delta_1, \delta_2) > \delta_1$. Let $\gamma_0 = T$-$\Delta_2\mathrm{cf}\,\beta = T$-$\Delta_2\mathrm{cf}\,\beta^*$ and h_1, h_2 be order-preserving T-Δ_2 functions mapping γ_0 unboundedly into β^*, β. Also, let f: $S_\beta \to \beta^*$ be 1-1, β-recursive and $\{f^\sigma\}$ be an enumeration of Graph(f).

Define $p(\alpha) = c(h_1(\delta), f(\alpha))$ where $\delta = \mu\delta[\alpha \in S_{h_2(\delta)}$ and $f^{h_2(\delta)}(\alpha)$ is defined].

p is 1-1 as c and f are. The $p'(\sigma, \alpha)$ of the lemma is the natural approximation to p using the approximations to h_1, h_2 and the enumeration $\{f^\sigma\}$.

Now if $\alpha \notin S_{h_2(\delta)}$, then for some $\delta' > \delta$, $p(\alpha) = c(h_1(\delta'), f(\alpha)) > h_1(\delta') > h_1(\delta)$. So $p^{-1}[h_1(\delta)]$ is $<_\beta$-bounded by $h_2(\delta)$. But also, $p(\alpha) \leqslant h_1(\delta)$ implies $f^{h_2(\delta)}(\alpha)$ is defined, so $p(\alpha) < h_1(\delta) \leftrightarrow p'(\sigma, \alpha) < h_1(\delta)$ for all $\sigma \geqslant_\beta h_2(\delta)$. ⊣

We use p to define the priority listing. Of course, at stage σ we use the approximation $p'(\sigma, \alpha)$ to $p(\alpha)$. Let L: $S_\beta \to S_\beta$ be the enumeration defined by

$$L(z) = \begin{cases} x, & z = \langle x, y\rangle \text{ for some } y, \\ 0, & \text{otherwise.} \end{cases}$$

Stage σ, $L(\sigma) = \langle 0, e\rangle$. We consider S_e. If $e \cap A^{<\sigma} \neq \varnothing$ or $e \nsubseteq \beta^*$, go to the next stage. Otherwise let x be the least member of e such that $x > p'(\sigma, \langle 0, e\rangle) + p'(\sigma, \langle 0, e\rangle)$ *and* x does not belong to any negative requirement for $T_{e'}$ where $T_{e'}$ has higher priority than S_e, i.e., $p'(\sigma, \langle 1, e'\rangle) < p'(\sigma, \langle 0, e\rangle)$. If such an x exists, set $A^\sigma = A^{<\sigma} \cup \{x\}$; if not, $A^\sigma = A^{<\sigma}$.

Stage σ, $L(\sigma) = \langle 1, e\rangle$. We consider T_e. If $e \subseteq A^{<\sigma}$ or $e \nsubseteq \beta^*$, go to the next stage. Otherwise, let x = least member of $e - A^{<\sigma}$ and set up the negative requirement for T_e of keeping x out of A.

If $L(\sigma) \neq \langle 0, e\rangle$ or $\langle 1, e\rangle$, go to the next stage. This ends the construction.

CLAIM 1. Let $\delta < \beta^*$ and let σ be as in 4.3(iii). Let $\sigma_0 >_\beta \sigma$ be the first stage such that $L[\{\sigma' | \sigma \leqslant_\beta \sigma' \leqslant_\beta \sigma_0\}] \supseteq \delta$. Then the activity of requirements S_e, T_e with $p(\langle 0, e\rangle)$, $p(\langle 1, e\rangle)$ less than δ ceases by stage σ_0.

PROOF. After stage σ_0, each such S_e, T_e has had the opportunity to act. The only reason for any of these requirements to act again is if $p(\sigma', \langle 0, e\rangle)$ or $p(\sigma', \langle 1, e\rangle)$ changes value for some $\sigma' \geqslant_\beta \sigma_0$, contradicting the choice of σ. $\dashv$

CLAIM 2. If $p(\langle 1, e\rangle) = \delta$, β-cardinality $(\delta) = \kappa$, then T_e is attempted at most κ-many times and is eventually satisfied.

PROOF. By Claim 1, there is a stage σ_0 beyond which all requirements $S_{e'}$ of higher priority than δ do not act. Then T_e will act (permanently) at most once beyond this stage. Since any two attempts at T_e before this stage are separated by the action of some $S_{e'}$ of higher priority, there are a total of at most κ-many attempts at T_e. $\dashv$

CLAIM 3. Each S_e is satisfied.

PROOF. Choose σ such that $p'(\sigma, \langle 0, e\rangle) = p(\langle 0, e\rangle)$, $L(\sigma) = \langle 0, e\rangle$. If o.t. $e = \beta^*$, $e \subseteq \beta^*$, then the x defined in the construction for stage σ must exist since the subset of β^* being kept out of A for the sake of requirements $T_{e'}$ of higher priority has size $< \beta^*$ by Claim 2. $\dashv$

CLAIM 4. $\beta^* - A$ has order type β^*.

PROOF. Just as in Claim 2 of the earlier simple set construction. $\dashv$

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