Vaught's Conjecture and Absoluteness

Let φ be a sentence of $L_{\omega_1\omega}$ and let $E(\varphi)$ denote the isomorphism relation on its countable models.

Then $E(\varphi)$ is analytic and for any analytic equivalence relation E we can define:

E is *perfect* if E has a perfect set of classes.

E is *scattered* if E does not have a perfect set of classes.

E is *trivial* if E has only countably many classes.

Question: Are these notions absolute?
Are they absolute when E is of the form $E(\varphi)$ for some φ?

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Proposition

"Perfect" is a Σ^1_2 property (of a code for E). Therefore both "perfect" and "scattered" (the negation of "perfect") are absolute.

Proof. E is perfect iff

There exists a perfect tree T such that for all $x \neq y$ in T, $\sim (xEy)$.

This is Σ^1_2. □

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Theorem

In L there is an analytic E which is scattered and nontrivial but becomes trivial after ω_1^L is made countable.

Proof. A *(Σ_ω-) master code* is a real coding the theory of some L_α.

The set of master codes is Π^1_1, as to be a master code is to satisfy an arithmetical property together with the assertion that the model described by the master code is wellfounded.

But now define:

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$$xEy \text{ iff } x, y \text{ are not master codes or } x = y.$$

This is analytic and has exactly ω_1^L classes.
If ω_1^L is collapsed then E becomes trivial. $\square$

However triviality is absolute for analytic equivalence relations of the form $E(\varphi)$, using Scott analysis.
I'll sketch how this works using the *Morley tree* $\mathcal{T}(\varphi)$ for φ.

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Suppose that $E(\varphi)$ is scattered (an absolute notion).

Let F_0 be a countable fragment of $L_{\omega_1\omega}$ containing φ.
By scatteredness there are only countably many F_0-types consistent with φ.

Let F_1 be a larger countable fragment containing the conjunctions of each of the F_0-types consistent with φ. Again, there are only countably many F_1-types consistent with φ.

Continue in this way for ω_1 steps, taking the fragment $F_{\alpha+1}$ to contain F_α and the conjunctions of the F_α-types consistent with φ, and F_λ the union of the F_α, $\alpha < \lambda$ for limit λ.

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Now that we have the fragments F_α, $\alpha < \omega_1$ we define the *Morley tree* $\mathcal{T}(\varphi)$ as follows:

At level 0 of the tree we place all complete F_0-theories containing φ.

At level $\alpha + 1$ of the tree we place all complete F_1-theories which extend a *non* $\aleph_0$*-categorical* theory on level α (i.e. a theory on level α with a non-atomic type).

At a limit level $\lambda < \omega_1$ we take the unions of all branches through the earlier levels. Again by scatteredness, this is still countable.

This completes the definition of the Morley tree.
Each countable model of φ is the unique model of some terminal node of $\mathcal{T}(\varphi)$. Thus φ is a counterexample to Vaught's Conjecture exactly if $\mathcal{T}(\varphi)$ has height ω_1.

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Also note that if we define

$$M \; E_\alpha \; N \text{ iff } M, N \text{ have the same } F_\alpha\text{-theory}$$

then $E(\varphi)$ (the isomorphism relation for countable models of φ) is the intersection of the Borel equivalence relations E_α. Thus $(E_\alpha \mid \alpha < \omega_1)$ is a *Burgess approximation* to $E(\varphi)$, i.e. a descending sequence of Borel equivalence relations with intersection $E(\varphi)$.

Now why is having only countably many models absolute?

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Proposition

The Burgess approximation $(E_\alpha \mid \alpha < \omega_1)$ *to* $E(\varphi)$ *is:*
(a) Σ_1*-definable, i.e. there is a* Σ_1 *function (in a fixed parameter coding* φ*) that takes a code for an ordinal* α *to a Borel code for* E_α*.*
And it is:
(b) Non-hesitating: If $E_\alpha = E_{\alpha+1}$ *then* $E_\alpha = E(\varphi)$*.*

(b) holds because $E_\alpha = E_{\alpha+1}$ implies that all models of theories on level α of the Morley tree satisfy the same F_α-types and therefore all such theories are $\aleph_0$-categorical.

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Hesitation of the master code example:

xEy iff x, y are not master codes or $x = y$

$xE_\alpha y$ iff x, y are not among the master codes in L_α or $x = y$

If α is the ω_1 of L_β then $E_\alpha = E_\beta$, so there are arbitrarily long hesitations in this Burgess approximation.

Question. Are we talking about model theory or more generally about Polish group actions?
I.e., if E is an analytic equivalence relation induced as the orbit equivalence relation of a Borel action of a Polish group on a Polish space, is triviality (having only countably many classes) absolute?

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One approach to this (suggested to me by André Nies) is to try to reduce an arbitrary orbit equivalence relation to a notion of equivalence on metric structures and then apply an analogous Scott/Morley analysis for metric structures.
This would be great and I hope it works!

Instead I will take a different approach, looking beyond orbit equivalence relations to analytic equivalence relations in general. Recall that orbit equivalence relations have only Borel classes.

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Definition

Let E be an analytic equivalence relation. Then E is weakly tame if there is a Σ^1_2 function f (in a parameter for E) such that for each x, $f(x)$ is a Borel code for the E-class of x.
E is tame if this holds absolutely, i.e. f has this property in all outer models as well.

Example (Sami, as modified by me):

xEy iff x, y compute the same master codes.

Then in L, E is weakly tame but not tame.
Moreover, E has only Borel classes of bounded rank.

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Theorem

(Becker) If E is an orbit equivalence relation (induced by a Borel action of a Polish group on a Polish space) then E is tame.

Theorem

Suppose that E is a tame analytic equivalence relation. Then triviality (having only countably many classes) for E is absolute. And E has a Σ_1-definable, non-hesitating Burgess approximation. In particular this holds for orbit equivalence relations.

And using a theorem of Stern:

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Theorem

Suppose that E is a tame analytic equivalence relation with classes of bounded Borel rank. Then E obeys Silver's dichotomy: either E has a perfect set of classes or only countably many classes.

This theorem follows from the previous one as Stern showed that the conclusion of this theorem holds for E (without tameness) after making $\aleph_{\omega_1}$ countable; then apply the previous theorem and absoluteness.

Commercial interruption

Before discussing the proofs of the above theorems, I pause to advertise the *generic Morley tree* $g\mathcal{T}(\varphi)$ and its use to give a new proof of:

Theorem

(Harrington) If φ is a counterexample to Vaught's Conjecture then φ has models in $\aleph_1$ of Scott ranks cofinal in ω_2.

Proof (Baldwin-SDF-Koerwien-Laskowski) Let $g\mathcal{T}(\varphi)$ be the Morley tree for φ in $V[G]$, where G is generic for collapsing ω_1 to ω. So the ω_1 of $V[G]$ is the ω_2 of V.

Commercial interruption, continued

By the scatteredness of φ, $g\mathcal{T}(\varphi)$ cannot depend on the choice of generic G so in fact $g\mathcal{T}(\varphi)$ is a tree in V of height ω_2^V. And for limit α, any model of a theory on the α-th level of $g\mathcal{T}(\varphi)$ has Scott rank at least α, so we need only show that the theories on the generic Morley tree $g\mathcal{T}(\varphi)$ do indeed have models in V.

Consider a pair (F, T) where T is on the generic Morley tree at some level α and $F = F_\alpha$, the fragment of $L_{\omega_2\omega}$ associated to that level. The theory T is *generically atomic*, i.e. atomic after F is made countable.

Commercial interruption, continued

Now using a chain of countable elementary submodels, we can write (F, T) as the direct limit of pairs $(\bar{F}_i, \bar{T}_i)$, $i < \omega_1$, where $\bar{F}_i$ is a countable fragment of $L_{\omega_1\omega}$ and $\bar{T}_i$ is an atomic theory in $\bar{F}_i$. Let M_i be the countable atomic model of $\bar{T}_i$. Then we have embeddings of model-fragment pairs

$$\pi_{ij} : (M_i, \bar{F}_i) \to (M_j, \bar{F}_j)$$

which are elementary in the sense that

$$M_i \models \psi(\bar{m}) \text{ iff } M_j \models \pi_{ij}(\psi)(\pi_{ij}(\bar{m})).$$

I.e., not only the model M_i but also the fragment $\bar{F}_i$ gets embedded by π_{ij}.

The direct limit of the countable models M_i $(i < \omega_1)$ is a model in $\aleph_1$ of our given theory T on the generic Morley tree, as desired. $\square$

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Now back to absoluteness.

Theorem

Suppose that E is a tame analytic equivalence relation. Then triviality for E is absolute.

Proof. Let f witness tameness.

If E has only countably many classes we can choose $\vec{x} = (x_n | n < \omega)$ such that each class contains x_n for some n and $\vec{c} = (c_n | n < \omega)$ so that $c_n = f(x_n)$ is a Borel code for the class of x_n for each n.

Thus if E has only countably many classes we have:

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$(*)$ There exist $\vec{x} = (x_n | n < \omega)$, $\vec{c} = (c_n | n < \omega)$ and $\alpha < \omega_1$ such that:

1. For each n, $c_n = f(x_n)$ is a Borel code for the E-class of x_n of Borel rank $< \alpha$.
2. If B_{c_n} is the Borel set coded by c_n then the B_{c_n}'s cover the space.

Conversely, if $(*)$ holds then as the B_{c_n}'s in $(*)$ are the E-classes of the x_n's and cover the space, E has only countably many classes.

Finally, $(*)$ is Σ^1_2 and therefore absolute. $\square$

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Now that we know that having countably many classes is Σ^1_2, we can produce a Σ_1-definable, non-hesitating Burgess approximation for E.

Begin with the representation

$$xEy \text{ iff } T(x,y) \text{ is illfounded}$$

and define relations

$$R_\alpha(x,y) \text{ iff } T(x,y) \text{ has rank at least } \alpha.$$

There is a Σ_1-definable function that produces a Borel code for R_α from a code for α. Burgess showed that R_α is an equivalence relation E_α for unboundedly many α, and by absoluteness it follows from this that any analytic equivalence relation E has a Σ_1-definable Burgess approximation $(E_\alpha \mid \alpha < \omega_1)$.

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Now we define a non-hesitating, Σ_1-definable Burgess approximation $(E'_\alpha \mid \alpha < \omega_1)$:

If E has only countably many classes then by absoluteness this is the case in L and so $(*)$ above will hold for some L-countable α; in this case we can take the trivial Burgess approximation $E'_\alpha = E$ for all α, as E has a Σ_1-definable Borel code.

Otherwise, set $E'_0 = E_0$, and if E'_α is defined choose $E'_{\alpha+1}$ to be E_β where β is least so that E_β is properly contained in E'_α. By absoluteness this least β is less than $\omega_1^{L[c]}$ for any code c for α and therefore we can compute a Borel code for $E'_{\alpha+1}$ via a Σ_1 function applied to c. The resulting Burgess approximation is Σ_1-definable and non-hesitating, as desired.
(*Remark:* This can be made uniform.) $\square$

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Some final remarks and questions

1. Sami showed that an orbit equivalence relation with classes of bounded Borel rank is in fact Borel; Gao showed that this is not true for arbitrary analytic equivalence relations, even when all classes have size at most 2.
2. There are nontrivial, scattered analytic equivalence relations with only Borel classes, with both Borel and non-Borel classes and with only non-Borel classes.
3. As the triviality of an analytic equivalence relation is a Σ^1_3 property, it is consistent relative to a reflecting cardinal (between inaccessible and Mahlo) that triviality is set-generically absolute for all analytic equivalence relations. Is an inaccessible enough for this? And is it consistent that triviality for analytic equivalence relations is absolute for class-forcing?

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4. Let $\equiv_\alpha$ be elementary equivalence for sentences of rank less than α. Suppose that $\equiv_\alpha$ equals $\equiv_{\alpha+1}$ on models of φ, for some α. Must $\equiv_\alpha$ equal isomorphism on models of φ? This is the case if φ does not have a perfect set of models.

And finally:

Is Vaught's Conjecture absolute?

It is consistent that it is set-generically absolute (as it is a Σ^1_3 statement), but can one rule out that $0^\#$ is the least *L*-degree of a counterexample?