The Outer model programme and Quasi lower bounds

Let LC denote a large cardinal property

Two important Jensen programmes:

A. Inner model programme. Show that models which witness LC have *L-like* inner models which also witness LC.

B. Core model programme. Define an L-like inner model K such that if K is a *bad approximation* to V then there is an inner model which witnesses LC

Consequences:

Inner model programme:
$\mathrm{Con}(\mathrm{ZFC}+\mathrm{LC}) \rightarrow \mathrm{Con}(\mathrm{ZFC}+\mathrm{LC}+\varphi)$ (φ an L-like property)
Core model programme:
$\mathrm{Con}(\mathrm{ZFC}+\varphi) \rightarrow \mathrm{Con}(\mathrm{ZFC}+\mathrm{LC})$ (consistency lower bounds)

The Outer model programme and Quasi lower bounds

The Jensen programmes are *difficult*

In this talk we discuss two easier alternatives

A. Outer model programme.* Show that models which witness LC have *L-like* outer models which witness LC

The Outer model programme is not new
First major result was proved by ... Jensen!
Con(ZFC+ a Ramsey) $\rightarrow$ Con(ZFC+ a Ramsey +GCH)

The first proof of this using inner models came much later (Dodd-Jensen)

B. Quasi lower bound programme*

This is new, combines A* with work of Neeman

The Inner model programme

A. Obtaining L-like models with Large Cardinals via the Inner model programme

Example 1: Inaccessible cardinals
Easy: If κ is inaccessible, then $L \models \kappa$ inaccessible.

Example 2: Measurable cardinals
Scott: $L \models$ There is no measurable cardinal
What inner model shall we use?

The Inner model programme

Relativised L: $\mathcal{L}^E_\alpha = (L^E_\alpha, \in, E_\alpha)$, $\alpha \in \mathrm{Ord}$

$\mathcal{L}^E_0 = (\emptyset, \emptyset, \emptyset)$
$\mathcal{L}^E_{\alpha+1} = (\mathrm{Def}(\mathcal{L}^E_\alpha), \in, E_{\alpha+1})$ (in fact $E_{\alpha+1} = \emptyset$)
$\mathcal{L}^E_\lambda = (\bigcup_{\alpha<\lambda} L^E_\alpha, \in, E_\lambda)$

Desired inner model is $L[\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle] = L[E]$, where the nonempty E_α's are embeddings

The Inner model programme

Theorem

Suppose that there is a measurable cardinal. Then there exists $E = (E_\alpha \mid \alpha \in Ord)$ such that:

1. For limit λ, E_λ is either empty or an embedding $E_\lambda : L^E_\alpha \to L^E_\lambda$ for some $\alpha < \lambda$.

2. $L[E] \models$ There is a measurable cardinal.

3. $E \restriction \kappa$ is definable over $L_\kappa[E]$ uniformly for infinite cardinals κ.

4. Condensation: With some restrictions, $M \prec \mathcal{L}^E_\alpha$ implies M is isomorphic to some $\mathcal{L}^E_{\bar{\alpha}}$.

5. $L[E] \models \Diamond$, $\Box$ and (gap 1) Morass

3 $\longrightarrow$ locally definable wellordering

4 $\longrightarrow$ GCH

This Theorem has been generalised after great effort to stronger large cardinal properties.

The Inner model programme

Why is the Inner Model Program so difficult?

Condensation: $M \prec \mathcal{L}^E_\alpha = (L^E_\alpha, \in, E_\alpha)$ implies M is isomorphic to some $\mathcal{L}^E_{\bar\alpha} = (L^E_{\bar\alpha}, \in, E_{\bar\alpha})$.

With Gödel's methods, M is isomorphic to some $\mathcal{L}^F_{\bar\alpha} = (L^F_{\bar\alpha}, \in, F_{\bar\alpha})$

Goal: $\mathcal{L}^F_{\bar\alpha} = \mathcal{L}^E_{\bar\alpha}$

Only known technique: *Comparison method*

The Inner model programme

Let $\bar{M}$, $\bar{N}$ denote $\mathcal{L}^F_{\bar{\alpha}}$, $\mathcal{L}^E_{\bar{\alpha}}$. Construct chains of embeddings

$$\bar{M} = \bar{M}_0 \to \bar{M}_1 \to \bar{M}_2 \to \cdots \to \bar{M}_\lambda$$
$$\bar{N} = \bar{N}_0 \to \bar{N}_1 \to \bar{N}_2 \to \cdots \to \bar{N}_\lambda$$

until $\bar{M}_\lambda = \bar{N}_\lambda$. Then conclude that $\bar{M} = \bar{N}$.

The embeddings come from *iteration*.

Key question: Is $\bar{M}$ iterable, i.e., are the models $\bar{M} = \bar{M}_0 \to \bar{M}_1 \to \bar{M}_2 \to \cdots \to \bar{M}_\lambda$ well-founded?

Iterability problem. Assuming the existence of large cardinals, show that there are iterable structures $M = (L^E_\alpha, \in, E_\alpha)$ which contain large cardinals.
Solved only up to the level of Woodin cardinals

The Outer model programme

Obtaining L-like models with Large cardinals via the Outer model programme.

Theorem

Suppose that there is a superstrong cardinal. Then there exists an outer model $L[A]$ of V (obtained by forcing) such that:

1. *A is a class of ordinals.*
2. *$L[A] \models$ There is a superstrong cardinal.*
3. *(with David Asperó) $A \cap \kappa^+$ is uniformly definable over $L_{\kappa^+}[A]$ for regular cardinals $\kappa > \omega$.*
4. *(with Peter Holy) Condensation: With some restrictions, $M \prec (L_\alpha[A], \in, A \cap \alpha)$ implies M is isomorphic to some $(L_{\bar{\alpha}}[A], \in, A \cap \bar{\alpha})$.*
5. *$L[A] \models \Diamond$, $\Box$ and (gap 1) Morass*

What is a superstrong cardinal?

The Outer model programme

Suppose $j : V \longrightarrow M$.
Critical point of j = least ordinal κ such that $j(\kappa) \neq \kappa$.
j is α-strong iff $V_\alpha \subseteq M$
Superstrong = $j(\kappa)$-strong
Hyperstrong = $j(\kappa) + 1$-strong
n-superstrong = $j^n(\kappa)$-strong
ω-superstrong = $j^\omega(\kappa)$-strong
$(j^\omega(\kappa) + 1)$-strong is inconsistent!
ω-superstrong is at the edge of inconsistency

κ is n-superstrong iff j is n-superstrong
(similarly for hyperstrong, ω-superstrong)

The Outer model programme

Are there L-like models past a superstrong?

Jensen: Subcompact $\rightarrow \square$ fails
(subcompact is between superstrong and hyperstrong)

Theorem

With $\square$ omitted, the previous Theorem (stated for superstrong) also holds for ω-superstrong.

Conclusion:
L-like is consistent with superstrong
L-like without $\square$ is consistent with all large cardinals

The Core model programme

B. Core model programme. Define an L-like inner model K such that if K is a *bad approximation* to V then there is an inner model witnessing LC

Equivalently, K should have the following property:

If there is no inner model with a certain large cardinal property then K is a *good approximation* to V

Examples of "good approximation":

Covering: Many sets of ordinals in V are contained in sets in K of the same size
Weak covering: κ^+ of V equals κ^+ of K for many cardinals κ
Rigidity: There is no nontrivial elementary embedding from K to K

The Core model programme

Example: Dodd-Jensen core model K_{DJ} for a measurable

GCH and $\square$ hold in K_{DJ}
If there is no inner model with a measurable then Covering holds for K_{DJ}

Conclusion: If GCH fails at a singular strong limit cardinal or $\square$ fails at a singular cardinal then there is an inner model with a measurable

Problem: The core model programme is even more difficult than the inner model programme!

Quasi lower bounds

A possible alternative:

B. Quasi lower bounds*

Motivating example: Neeman-Schimmerling work on PFA fragments

PFA = Proper forcing axiom = Martin's axiom for proper forcings

Baumgartner: Con(ZFC+ a supercompact) $\rightarrow$ Con(ZFC + PFA)

Popular Conjecture: The converse holds
(beyond current core model techniques)

Quasi lower bounds

A forcing P is κ-*linked* iff it is the union of κ-many pairwise compatible subsets

Theorem

(Neeman-Schimmerling) (a) Con(ZFC+ a Σ^2_1 indescribable) $\rightarrow$ Con(ZFC + PFA for c-linked forcings).
(b) More generally, Con(ZFC+ a Σ^2_1 indescribable n-gap) $\rightarrow$ Con(ZFC + PFA for c^{+n}-linked forcings).

Using L, Neeman obtained a consistency lower bound for (a):

Con(ZFC + PFA for c-linked forcings) $\rightarrow$
Con(ZFC+ a Σ^2_1 indescribable)

Quasi lower bounds

[(Neeman-Schimmerling) (a) Con(ZFC+ a Σ^2_1 indescribable) $\rightarrow$ Con(ZFC + PFA for c-linked forcings). (b) More generally, Con(ZFC+ a Σ^2_1 indescribable n-gap) $\rightarrow$ Con(ZFC + PFA for c^{+n}-linked forcings).]

Neeman and Schimmerling conjecture the following:

Con(ZFC + PFA for c^+-linked forcings) $\rightarrow$
Con(ZFC+ a Σ^2_1 indescribable 1-gap)

Problem: Σ^2_1 indescribable 1-gaps have consistency strength beyond a superstrong, and therefore are beyond the reach of current core model theory!

Quasi lower bounds

Neeman obtained a partial result:

Theorem

(Neeman) Suppose that V is a "fine structural model" and PFA for c^+-linked forcings holds in a proper forcing extension of V. Then in V there is a Σ^2_1 indescribable 1-gap.

V is Neeman fine-structural iff V is built from extenders, the extender-hierarchy on V satisfies enough condensation and is acceptable, and V satisfies enough of Jensen's $\square$ principle

Problem: Are there any Neeman fine-structural models with a Σ^2_1 indescribable 1-gap? Are there any with large cardinal properties beyond Woodin cardinals?

Quasi lower bounds

Observation (Peter Holy): Extenders are irrelevant to Neeman's proof; one only needs enough of Jensen's $\square$ principle and *some* hierarchy on V which satisfies enough condensation and is acceptable.

Call such models *sufficiently L-like*

Now we invoke the techniques of the outer model programme:

Theorem

(F-Holy) Suppose that there is an ω-superstrong cardinal in V. Then some forcing extension of V is both sufficiently L-like and contains an ω-superstrong cardinal.

(Enough $\square$ is easy, enough condensation is harder, acceptability is the hardest)

Quasi lower bounds

Now we obtain the following quasi-lower bound result.

Corollary

(F-Holy) It is consistent with a proper class of subcompact cardinals that PFA for c^+-linked forcings fails in all proper forcing extensions.

(Subcompacts are a little bit weaker than Σ^2_1 indescribable 1-gaps.)

Thus for all practical purposes, PFA for c^+-linked requires more than subcompacts; this is a *quasi lower bound* result

Conclusion:

Consistency lower bounds need Core model theory
Consistency quasi lower bounds may only need Outer model theory

Questions

A. Inner model theory
Assume that there is a superstrong. Is there an inner model satisfying GCH with a superstrong?

A*. Outer model theory
Is it consistent with a superstrong to have a definable wellorder of $H(\lambda^+)$ for all singular λ?

B. Core model theory
Does the failure of $\square$ at a singular cardinal imply the existence of an inner model with a superstrong?

B*. Quasi lower bounds
Is it consistent with a superstrong that $\square$ holds at all singular cardinals in all (proper) forcing extensions?

I congratulate Ronald on the occasion of this excellent meeting!