Criteria for the Choice of New Axioms

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1. Pure and Applied Set Theory

I see set theory as consisting of two overlapping areas, the *pure* and the *applied*. In *pure* set theory we are trying to find justifiable ways to strengthen the ZFC axioms (syntactic), and correspondingly, justifiable pictures of the set-theoretic universe (semantic). I use the phrase *applied set theory* to refer to everything else.

According to this definition, the Martin-Steel Theorem is applied, rather than pure. The Covering Theorem is a result of pure set theory, with hundreds of applications.

The most important results in pure set theory were obtained by trying to solve problems in applied set theory. For example, Jensen proved the Covering Theorem in order to better understand the singular cardinal problem.

The converse is not true. Shelah's work provides a clear counterexample. His interest is to obtain results in or consistency results relative to ZFC + large cardinals, rather than to examine the nature of inner models of set theory, or what the sources of large cardinals might be.

The rest of this article is concerned exclusively with pure set theory.

2. Extending ZFC

What should we expect from an extension of ZFC? Obviously we cannot hope to "prove" the new axioms we choose, and by Gödel we cannot even hope to prove their consistency.

Desirable properties for new axioms are the following.

Naturality: The axioms should come directly from the semantics of ZFC, and constitute an attempt to clarify the structure of the set-theoretic universe V.

Power: The axioms should explain a lot.

Stability: The axioms should be unaffected by small changes, and in particular, small changes should not knowingly lead to inconsistency. Semantically, a small extension of a model of the axioms (obtained by set-forcing or by a reasonable class-forcing) should also be a model.

A number of interesting new axioms have arisen out of applied set theory over the past several decades. Recently there have been suggestions that some of these axioms provide us with the "right" extension of ZFC. Unfortunately these suggestions suffer from oversimplification, and lead to axioms that violate the above criteria.

Remark. My aim here is not to argue for or against the *consistency* of various axioms; I simply assume that the axioms that we do not currently know to be inconsistent are in fact consistent. Instead, I am discussing the *appropriateness* of axioms, based upon the above criteria.

3. Examples

a. V=L

Of course this axiom is natural and very powerful. But by the work of Cohen we know that the axiom

$$V = L[\text{a Cohen Real}]$$

is just as consistent, and surely constitutes a small change to the axiom $V = L$. Therefore the criterion of stability is violated. The same problem exists with any axiom of the form

$$V = L[G] \text{ where } G \text{ is } P\text{-generic over } L$$

for any L-definable set-forcing P, as one can similarly violate this by forcing with even larger partial orders.

b. Large cardinals

Typically these are of the form

$$\text{There exists } j : V \to M, \text{ where } M \text{ is "close" to } V.$$

Certainly such axioms are natural and very powerful. They are however unstable: If we require $M = V$, we have a contradiction. If we only require

M to agree with V up to $j(\kappa)$ where κ is the critical point of j, then by stability, we should also allow agreement up to arbitrary iterates of j applied to κ, another contradiction. A possibility is to allow M to agree with V up to $j(f)(\kappa)$ for any particular function $f : \kappa \to \kappa$, but not simultaneously for all such f; more about this later.

c. Determinacy

Of course I am not referring to the full axiom AD, as this contradicts the axiom of choice, but rather to determinacy for sets of reals that are definable (say, ordinal-definable with real parameters). This axiom has proved to be powerful and stable. Unfortunately the existence of strategies for infinite games does not arise naturally out of the semantics of ZFC.

However I will argue later that some definable determinacy is a consequence of natural axioms, even though determinacy itself does not qualify as one.

d. Absoluteness principles

These are principles which assert that the truth of certain formulas is not affected by enlarging the universe in certain ways. The classical example of this is Shoenfield absoluteness, which says that $\Sigma_1(H(\omega_1))$ formulas (with parameters) are absolute for arbitrary extensions.

Absoluteness principles are however unstable. Absoluteness for $\Sigma_2(H(\omega_1))$ formulas cannot hold with respect to all (ω_1-preserving) class-forcing extensions. Even Σ_1 absoluteness cannot hold for $H(\omega_2)$ with respect to all (ω_1- and ω_2-preserving) set-forcing extensions, nor with respect to $H(c^+)$ with respect to ccc forcing extensions.

e. Forcing axioms

The most common such axioms assert that for certain forcings P and certain collections X of dense subsets of P, there is a compatible subset G of P which intersects all elements of X. The classical example is Martin's axiom (at ω_1), which asserts this for ccc P and collections X of cardinality ω_1.

As with the absoluteness principles, these axioms suffer from instability: One cannot have this forcing axiom for ω_1-many dense sets with respect to all ω_1-preserving set-forcing extensions or for ω_2-many dense sets with respect to all ω_1- and ω_2-preserving set-forcing extensions.

Other types of forcing axioms have also been considered. Foreman, Magidor and Shelah considered the statement:

Every set-forcing either adds a real or collapses a cardinal.

Unfortunately, little is known about this axiom.

Chalons (as modified by Larson) proposed:

"If a statement with real parameters holds in a set-forcing extension and all further set-forcing extensions, then it holds in V; moreover this property is not only true in V, but also in all set-generic extensions of V."

Woodin proved the consistency of this axiom from large cardinals. Unfortunately, even a weak form of this axiom is inconsistent when "set-forcing" is replaced by "class-forcing", in violation of stability. A reasonable class-forcing version of this axiom is not known.

Yet another kind of forcing axiom will be discussed below.

f. Strong logics

These are logics whose set of validities is large and remains unchanged by set-forcing. One can obtain such a logic as follows: Say that φ is $**$-provable iff for some set-forcing P, if P belongs to V_α and V_α satisfies ZFC, then V_α^{P*Q} satisfies φ for all Q in V_α^P. Woodin proposes the use of such a strong logic, together with the existence of a proper class of Woodin cardinals. This gives a $**$-complete theory of $H(\omega_1)$ and, assuming that $H(\omega_2)$ is obtained by forcing with Woodin's forcing P_{max} over $L(R)$, gives a $**$-complete theory of $H(\omega_2)$. Therefore under Woodin's assumptions, the theory of $H(\omega_2)$ cannot be changed by set-forcing.

There are several difficulties with this approach.

i. The assumption of the existence of a proper class of Woodin cardinals is left unjustified. However I will suggest below an argument in favour of an *inner model* for this assumption.
ii. Although strong logics are immune to set-forcing, they are not immune to class-forcing. Class-forcing methods provide consistent ways to enlarge the set-theoretic universe, in the same way that set-forcing methods do. Therefore adopting as new axioms the validities of a logic with only set-generic absoluteness violates stability.
iii. The axiom asserting that $H(\omega_2)$ is obtained by set-forcing over $L(R)$ is easily contradicted by class-forcing, and therefore unstable.

A strong logic whose validities are absolute for (appropriate) class-forcing is not known.

4. Patience and Necessity

The most important axioms that have been explored until now have arisen naturally and necessarily out of the need to solve central problems in applied set theory. This is especially true of the large cardinal axioms, which have even provided a measure for the consistency strength of virtually all set-theoretic statements. But in my view we should not impatiently assert that these axioms are "correct" until we can derive them from other axioms which meet strict criteria like the ones discussed above.

Ideally, we could aim for the following.

Necessity: The axioms should be necessary, in the sense that their failure lead to an unacceptable picture of the set-theoretic universe.

Necessity is very strong. It implies uniqueness: any two necessary axioms must be compatible with each other. I do not know how strong a necessary extension of ZFC can be. It may turn out that uniqueness fails, and that there are mutually contradictory extensions of ZFC, each of which provides a natural, strong and stable description of (parts of) the set-theoretic universe.

In my view, the correct axioms for the first-order theory of $H(\omega)$ are provided by finite set theory. Surely these axioms are necessary, and in my view they are sufficient, as $H(\omega)$ is the unique well-founded model of this theory and no clear examples of ill-founded models are known. I also believe that the correct axioms for the first-order theory of $H(\omega_1)$ are provided by PD (projective determinacy). Below I will provide an argument for the necessity of inner models with Woodin cardinals, and therefore of this theory. However I have not seen a convincing argument that PD is sufficient, in the sense that it captures the full first-order theory of $H(\omega_1)$ (although I do believe this to be the case).

Remark. Necessity may require a modification to our earlier criterion of Stability. The reason is that we do not yet know if Necessity leads to unstable axioms. The modified form of Stability would say: The axioms should be unaffected by small changes, unless they are necessary and a small change leads to inconsistency. None of the examples considered earlier were of necessary axioms, and therefore those which were unstable in the old sense remain so in this modified sense.

To discover necessary axioms, we can only begin with the basic techniques that we have for forming well-founded models of set theory, due to Gödel and Cohen. So I begin by considering L and its forcing extensions.

We have seen that by stability, the universe must contain nonconstructible sets, and indeed such sets which are P-generic over L for various constructible forcings P. A natural question to ask is:

Which constructible forcing notions P have generics (over L)?

Stability requires that if P has a generic in a small extension of V then it already has one in V. If P is countable then a P-generic extension of V is a small extension and therefore we necessarily have:

$(*)$ V is *L-saturated for countable forcings*: If P is a countable constructible forcing then P has a generic.

This axiom is not very strong; indeed it holds in L[a Cohen real].

Next I consider L-saturation for ω_1-forcings, i.e., forcings with universe ω_1. We consider only extensions which preserve the notion "constructible ω_1-forcing", i.e., extensions which preserve ω_1. Which such extensions shall we take to be the "small" extensions? Surely any set-generic extension should qualify and therefore we have:

$(**)$ V is *L-saturated for ω_1-forcings*: If P is a constructible ω_1-forcing with a generic in an ω_1-preserving set-generic extension of V then P has a generic in V.

Theorem 1. The following are equivalent:
(a) $(**)$ holds.
(b) $(**)$ holds in the stronger form: If P is a constructible ω_1-forcing with a generic in an arbitrary ω_1-preserving extension of V then P has a generic in V.
(c) $0^\#$ exists.

The existence of $0^\#$ is therefore in my view necessary, as otherwise V is not saturated for constructible ω_1-forcings, a violation of stability.

Can we go further? Obviously we can repeat what we have just done for the model $L[0^\#]$, and obtain the existence of $0^{\#\#}$. By iterating further we get $0^{\#^n}$ for any n and even $0^{\#^\omega}$. Can we continue this sequence long enough to reach a measurable cardinal?

Rather than iterate #'s, we can reach our goal directly, by generalising saturation from L to larger inner models. Suppose that M is an inner model, defined by the formula φ. How shall we define M-saturation for ω_1-forcings? In the case $M = L$ it was important to only consider extensions of V which preserve the concept "constructible ω_1-forcing". For general M, we must be careful to guarantee that not only ω_1, but also the interpretation of φ, the defining formula for M, does not change.

Definition. Let φ define the inner model M. We say that V is *M-saturated for ω_1-forcings (via φ)* iff whenever an ω_1-forcing in M has a generic (over M) in an ω_1-preserving extension W of V where $\varphi^W = M$, it already has a generic in V.

Now we apply this to the Dodd-Jensen core model K_{DJ}, using the standard defining formula for this model (whose interpretation is unchanged by set-forcing).

Theorem 2. (a) Suppose that V is K_{DJ}-saturated for ω_1-forcings. Then there is an inner model with a measurable cardinal.
(b) Conversely, suppose that there is an inner model with a measurable cardinal κ, where κ is countable. Then V is K_{DJ}-saturated for ω_1-forcings.

If we apply saturation for ω_1-forcings to Mitchell's core model, we can obtain measurable cardinals of higher order. Further strength appears however to require the use not only of larger inner models, but also of larger forcings. I can only offer a conjecture about this.

If M is an inner model defined by a formula φ and α is an uncountable ordinal, then we say that V is *M-saturated for α-forcings (via φ)* iff whenever an α-forcing in M has a generic (over M) in an extension W of V where V, W have the same cardinals $\leq \alpha$ and $\varphi^W = M$, it already has a generic in V.

A *Woodin cardinal* is a cardinal κ such that for each $f : \kappa \to \kappa$ there is an elementary embedding $j : V \to M$ with critical point κ and $V_{j(f)(\kappa)} \subseteq M$. Under appropriate assumptions (e.g., if there is no inner model with a Woodin cardinal, or if every set belongs to an inner model with a Woodin cardinal and a measurable above), and assuming a strong enough class theory (Ord is "subtle"), Steel constructs an inner model K_S which does for one Woodin cardinal what Silver's L_μ does for one measurable cardinal. As with the core models mentioned earlier, K_S is defined by a formula which has the same interpretation in all set-generic extensions of V.

Theorem 4. Suppose that V is K_S-saturated for $\aleph_\omega^{+K_S}$-forcings. Then there is an inner model with a Woodin cardinal.

Unfortunately, in Theorem 4 saturation is applied to forcings which destroy CH and in fact which add $\aleph_\omega^{+K_S}$ reals. I conjecture that Theorem 4, and a suitable converse, hold even when saturation is restricted to GCH-preserving forcings.

Conjecture 5. (a) Suppose that V is K_S-saturated for $\aleph_\omega^{+K_S}$-forcings which preserve GCH over K_S. Then there is an inner model with a Woodin cardinal. (b) Conversely, suppose that every set belongs to an inner model with a Woodin cardinal and a measurable above. Then V is K_S-saturated for $\aleph_\omega^{+K_S}$-forcings which preserve GCH over K_S.

Conjecture 5 yields the necessity of inner models with Woodin cardinals. Stability then implies that for each n there are inner models with n Woodin cardinals containing any given real, and therefore that PD holds.

CUB-Completeness

There is another type of necessary axiom which, although not as natural as forcing-saturation, does provably lead to inner models with Woodin cardinals. Suppose that M is an inner inner model defined by the formula φ and κ is a regular uncountable cardinal. We say that V *is CUB-complete over M at κ (via φ)* iff whenever $A \subseteq \kappa$ belongs to M and has a CUB subset in an extension W of V where $\varphi^W = M$ and all cardinals $\leq \kappa$ are preserved, then A already has as CUB subset in V.

Theorem 6. (a) For any regular uncountable cardinal κ, V is CUB-complete over L at κ iff $0^{\#}$ exists.
(b) If V is CUB-complete over K_{DJ} at ω_1 then ω_1 is measurable in an inner model; conversely, if κ is a cardinal and some ordinal $\alpha < \kappa$ is measurable in an inner model, then V is CUB-complete over K_{DJ} at κ.
(c) For $\kappa > \aleph_\omega$, if V is CUB-complete over K_S at κ then there is an inner model with a Woodin cardinal. Conversely, if every set belongs to an inner model with a Woodin cardinal and a measurable above, then V is CUB-complete over K_S at every regular κ.

Obtaining further strength from CUB-completeness is obstructed only by the current failure of core model theory to reach very far past Woodin cardinals.

The above approaches, forcing-saturation and CUB-completeness, though they justify the existence of *inner models* with Woodin cardinals, do not justify the existence of Woodin cardinals in V. Indeed, I am not optimistic about the possibility of finding good arguments for the existence of large cardinals in V until a good criterion is found for excluding those class-forcings which destroy large cardinal properties. Fortunately, large cardinals in V do not appear to be necessary to reach the right axioms for $H(\omega_2)$, a goal which in my view is still well beyond our reach.