Selected Topics in Logic
Wintersemester 2012

1.-2.Vorlesungen

Introduction

In Barcelona during 2009-2011 I ran a project called the "Infinity Project". The idea was to find new results in logic through collaborations between people in its different subfields: computability theory, model theory, proof theory and set theory. It worked rather well, and what I want to do in this course is report on what we discovered, as well as some further results in the same spirit.

The Infinity Project was divided into themes as follows:

Computations and Models
Computations and Proofs
Computations and Sets
Models and Sets
Proofs and Sets

You'll notice that "Models and Proofs" is missing; that's because we had enough to do with the other 5 topics.

Topic 1. Proofs and Sets: Forcing and Provably Recursive Functions

In set theory one has a "ground model" with a given set of functions from ω to ω. Then one of three things can happen with regard to eventual domination (mod finite) when passing to a larger model, as exemplified by:

1. Sacks forcing: One adds new functions but any new function is dominated by an old (ground model) function.
2. Cohen forcing: One adds a new function that cannot be dominated by a ground model function but no single function which dominates all ground model functions. If one adds two such functions f, g using "Cohen $\times$ Cohen" forcing, then in addition any function added by both f and g is in the ground model.
3. Hechler forcing: One adds a new function that dominates all ground model functions. If one adds two such functions f, g using "Hechler $\times$ Hechler"

forcing, then again any function added by both f and g is in the ground model.

The analogy in proof theory is the following: Fix a theory T like PA. Let us take the "T-provably recursive" functions to be those with primitive recursive graph (the honest functions) such that for some choice of primitive recursive representation of that graph, totality of the function is T-provable.

Theorem 1. *There exists an honest recursive function f such that f is not provably recursive in PA (via any primitive recursive representation of its graph) and such that no g which is provably recursive in PA + Tot(f) (where f is expressed using any primitive recursive graph representation and Tot(f) expresses the fact that f is total via this representation) dominates all provably recursive functions of PA.*

Theorem 1 is a proof-theoretic analogue of Cohen forcing.

Theorem 2. *There are total functions f_0, f_1 with primitive recursive graphs which are not provably recursive in PA (via any primitive recursive graph representation), yet any function which is provably recursive in both PA + Tot(f_0) and PA + Tot(f_1) (where these are expressed using primitive recursive graph representations) is in fact provably recursive in PA.*

Theorm 2 is a proof-theoretic analogue of Cohen forcing $\times$ Cohen forcing.

I'll sketch the proof of Theorem 1, borrowing some facts from the proof theory of PA. I won't prove Theorem 2, whose proof is similar but more difficult.

Proof of Theorem 1. We need to use the Hardy functions, defined by:

$H_0(n) = n$
$H_{\alpha+1}(n) = H_\alpha(n+1)$
$H_\alpha(n) = H_{\alpha[n]}(n)$ for α limit

where α ranges over ordinals $\leq \varepsilon_0 = \omega^{\omega^{\omega^{\cdots}}}$ and $(\alpha[n] \mid n \in \omega)$ denotes the natural "fundamental sequence" converging to the limit ordinal α. The latter is defined as follows:

For an ordinal α such that $\alpha > 0$, α has a unique representation:

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k,$$

where $0 < k, n_1, \ldots, n_k < \omega$, and $\alpha_1 > \cdots > \alpha_k$.

For each limit ordinal $\lambda \leq \varepsilon_0$ we define a strictly monotone sequence $(\lambda[n] \mid n \in \omega)$ cofinal in λ. First assume that λ is less than ε_0 and write λ uniquely as

$$\lambda = \beta + \omega^\gamma \cdot m$$

where either $\beta = 0$ or β has normal form $\omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_l} \cdot m_l$ with $\beta_l > \gamma$.

Case 1. $\lambda = \beta + \omega^\gamma \cdot m$ and $\gamma = \delta + 1$.
Put $\lambda[n] = \beta + \omega^\gamma \cdot (m-1) + \omega^\delta \cdot (n+1)$.

Case 2. $\lambda = \beta + \omega^\gamma \cdot m$, and $\gamma < \lambda$ is a limit ordinal.
Put $\lambda[n] = \beta + \omega^\gamma \cdot (m-1) + \omega^{\gamma[n]}$.

Finally, if $\lambda = \varepsilon_0$ then we set $\varepsilon_0[0] = \omega$ and $\varepsilon_0[n+1] = \omega^{\varepsilon_0[n]}$.

We construct the desired f in stages. $d_0 = 0$. Assume we are at an even stage s and that d_s is defined. Assume that $f(x)$ is defined for $x < d_s$. Then set $d_{s+1} = H_{\varepsilon_0}(d_s)$ and extend f by $f(x) = H_{\varepsilon_0}(x)$ for $d_s \leq x < d_{s+1}$.

Now assume that we are at an odd stage s. Set $d'_{s+1} = H_{\varepsilon_0}(d_s)$ and $d_{s+1} = H_{\varepsilon_0}(d'_{s+1})$. We extend f by $f(x) = d'_{s+1} + x$ for $d_s \leq x < d_{s+1}$.

We need some important facts about the Hardy functions:

(a) If $\alpha \leq \beta \leq \varepsilon_0$ then H_α is dominated by H_β.
(b) Each H_α, $\alpha < \varepsilon_0$ is provably recursive in PA and any function provably recursive in PA is dominated by some H_α, $\alpha < \varepsilon_0$.
(c) If g is provably recursive in PA + Tot(f) then there is an $\alpha < \varepsilon_0$ such that for all x we have $g(x) < f^\alpha(x)$ where:

$$f^\alpha(x) = \max(\{f(x)\} \cup \{f^\beta(f^\beta(x)) : \beta < \alpha \wedge N\beta \leq f(N\alpha + x)\}).$$

Here $N\alpha$ is defined by $N0 = 0$ and $N\alpha = N\alpha_1 + \cdots + N\alpha_k + n_k$ if α has the normal form $\omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$.

Now since $f(x) = H_{\varepsilon_0}(x)$ for infinitely many x it follows from (b) that f is not provably recursive in PA.

Assume that $\mathrm{PA} + \mathrm{Tot}(f)$ proves $\mathrm{Tot}(g)$ and choose $\alpha < \varepsilon_0$ such that for all x we have $g(x) < f^\alpha(x)$ where $f^\alpha(x)$ is defined as in (c) above. Choose an odd stage s with $2 \cdot N\alpha + 12 \leq d'_{s+1}$. Then Weiermann shows that

$$f^\alpha(d'_{s+1}) \leq H_{\omega^{\alpha+\omega}+\alpha+10}(d'_{s+1}).$$

So the function $H_{\omega^{\alpha+\omega}+\alpha+10}$ is not eventually dominated by g. $\square$

Topic 2. Computations and Proofs: Slow Consistency

In computation theory the notion of Turing reducibility plays a central role. An important operation on the Turing degrees is the Turing jump, which provides a natural way of increasing any Turing degree to a larger one. Although there are many Turing degrees between $\mathbf{0}$ and $\mathbf{0}'$, exhibiting a "natural" such degree is a difficult task.

A proof-theoretic analog of the Turing jump is the consistency operator. For any consistent theory T obtained from PA by adding finitely many new axioms, the theory $T + \mathrm{Con}(T)$ is strictly stronger than T. Although there are many theories between PA and PA + Con(PA), exhibiting a "natural" such theory is a difficult task. I'll give here some examples using a notion of "slow consistency".

To motivate slow consistency I'll first discuss the interpretability of one theory in another. To simplify matters, we restrict attention to theories formulated in the language of PA which contain the axioms of PA and have a primitive recursive axiomatization, i.e. the axioms are enumerated by such a function. Let S and S' be such theories. We define what it means for S' to be *interpretable in* S (in symbols $S' \lhd S$). A *translation* is a function t from formulas of arithmetic to formulas of arithmetic such that for some fixed formulas $\eta_0(x), \eta_S(x, y), \eta_+(x, y, z), \eta_\times(x, y, z)$ and $\mu_t(x)$ we have:

$t(x = y) = (x = y)$
$t(x = 0) = \eta_0(x)$
$t(Sx = y) = \eta_S(x, y)$
$t(x + y = z) = \eta_+(x, y, z)$
$t(x \times y = z) = \eta_\times(x, y, z)$
$t(\sim \varphi) = \sim t(\varphi)$
$t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$
$t(\exists x \xi(x)) = \exists x(\mu_t(x) \wedge t(\xi(x)))$.

t is an *interpretation of* S' *in* S if S proves $t(\varphi)$ whenever S' proves φ. S' is *interpretable in* S iff there exists such an interpretation t.

For an integer $k \geq 0$, we denote by $T\restriction_k$ the theory consisting of the first k axioms of T. Let $\mathrm{Con}(T)$ be the arithmetized statement that T is consistent. A theory T is *reflexive* if it proves the consistency of all its finite subtheories, i.e. $T \vdash \mathrm{Con}(T\restriction_k)$ for all $k \in \mathbb{N}$.

For proofs of the next two results, see the book "Aspects of Incompleteness" by Lindström.

Theorem 3. *Any theory containing PA is reflexive.*

Another interesting relationship between theories is: $T_1 \subseteq_{\Pi^0_1} T_2$, i.e. every Π^0_1 theorem of T_1 is also a theorem of T_2.

Theorem 4. *Let S, T be theories containing PA with primitive recursive axiomatisations. Then:*

$$\begin{aligned} S \lhd T \quad &\text{if and only if} \quad \text{For all } n \in \mathbb{N},\ T \vdash \mathrm{Con}(S\restriction_n) && (1)\\ &\text{if and only if} \quad S \subseteq_{\Pi^0_1} T. && (2) \end{aligned}$$

Now we turn to slow consistency. We know that

$$\mathrm{Con}(\mathrm{PA}) \leftrightarrow \forall x\, \mathrm{Con}(\mathrm{PA}\restriction_x).$$

Given a function $f : \mathbb{N} \to \mathbb{N}$ (say provably total in PA) we are thus led to the following consistency statement:

$$\mathrm{Con}_f(\mathrm{PA}) = \forall x\, \mathrm{Con}(\mathrm{PA}\restriction_{f(x)}).$$

(It is perhaps worth pointing out that the exact meaning of $\mathrm{Con}_f(\mathrm{PA})$ depends on the representation that we choose for f.)

3.-4.Vorlesungen

Statements of the above form are interesting if the function f grows extremely slowly, has an infinite range but PA cannot prove that fact. To show this we need a modification of the Hardy hierarchy, called the fast-growing hierarchy:

$F_0(n) = n + 1$
$F_{\alpha+1}(n) = F_\alpha^{n+1}(n)$
$F_\alpha(n) = F_{\alpha[n]}(n)$ if α is a limit.

Like the H_α's, the F_α's are defined for all $\alpha \leq \varepsilon_0$. The exact relationship between the two hierarchies is given by: $F_\alpha = H_{\omega^\alpha}$. In particular, $F_{\varepsilon_0} = H_{\varepsilon_0}$.

Definition 5. *Define*

$$F_{\varepsilon_0}^{-1}(n) \quad = \quad \max(\{k \leq n \mid \exists y \leq n \, F_{\varepsilon_0}(k) = y\} \cup \{0\}).$$

Note that the graph of $F_{\varepsilon_0}^{-1}$ has a Δ_0 definition. Thus it follows that $F_{\varepsilon_0}^{-1}$ is a provably recursive function of PA.

Let $\mathrm{Con}^*(\mathrm{PA})$ be the statement $\mathrm{Con}_{F_{\varepsilon_0}^{-1}}(\mathrm{PA})$; equivalently:

$$\forall x \, [F_{\varepsilon_0}(x) \downarrow \to \mathrm{Con}(\mathrm{PA} \restriction_x)].$$

Proposition 6. *PA* $\nvdash \mathrm{Con}^*(PA)$.

Proof: Aiming at a contradiction, suppose $\mathrm{PA} \vdash \mathrm{Con}^*(\mathrm{PA})$. Then $\mathrm{PA} \restriction_k \vdash \mathrm{Con}^*(\mathrm{PA})$ for all sufficiently large k. As $\mathrm{PA} \restriction_k \vdash F_{\varepsilon_0}(k) \downarrow$ on account of $F_{\varepsilon_0}(k) \downarrow$ being a true Σ_1 statement, we arrive at $\mathrm{PA} \restriction_k \vdash \mathrm{Con}(\mathrm{PA} \restriction_k)$, contradicting Gödel's second incompleteness theorem. $\square$

Proposition 6 holds in more generality.

Corollary 7. *If T is a recursive consistent extension of PA and f is a total recursive function with unbounded range, then*

$$T \nvdash \forall x \, \mathrm{Con}(T \restriction_{f(x)})$$

where $f(x) \downarrow$ is understood to be formalized via some Σ_1 representation of f.

The next goal is to show that $\mathrm{Con}(\mathrm{PA})$ is not derivable in $\mathrm{PA} + \mathrm{Con}^*(\mathrm{PA})$. We need some preparatory definitions.

Definition 8. *Let E denote the function $E(0) = 0$ and $E(n+1) = 2^{E(n)}$.*

Given two elements a and b of a non-standard model $\mathfrak{M}$ of PA, we say that b is much larger than a if for every standard integer k we have $E^k(a) < b$.

If $\mathfrak{M}$ is a model of PA and $\mathfrak{I}$ is a substructure of $\mathfrak{M}$ we say that $\mathfrak{I}$ is an initial segment of $\mathfrak{M}$, if for all $a \in |\mathfrak{I}|$ and $x \in |\mathfrak{M}|$, $\mathfrak{M} \models x < a$ implies $x \in |\mathfrak{I}|$. We will write $\mathfrak{I} < b$ to mean $b \in |\mathfrak{M}| \setminus |\mathfrak{I}|$. Sometimes we write $a < \mathfrak{I}$ to indicate $a \in |\mathfrak{I}|$.

Theorem 9. *Let $\mathfrak{N}$ be a non-standard model of PA, $a, e, c \in |\mathfrak{N}|$ be non-standard such that $\mathfrak{N} \models F_{\varepsilon_0}(a) = e$ and $\mathfrak{N} \models F_{\varepsilon_0}(a+1) = c$. Then for every standard n there is an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction.*

Definition 10. Below we shall need the notion of two models $\mathfrak{M}$ and $\mathfrak{N}$ of PA "agreeing up to e". For this to hold, the following conditions must be met:

1. e belongs to both models.
2. e has the same predecessors in both $\mathfrak{M}$ and $\mathfrak{N}$.
3. If d_0, d_1, and c are $\leq e$ (in one of the models $\mathfrak{M}$ and $\mathfrak{N}$), then $\mathfrak{M} \models d_0 + d_1 = c$ iff $\mathfrak{N} \models d_0 + d_1 = c$.
4. If d_0, d_1, and c are $\leq e$ (in one of the models $\mathfrak{M}$ and $\mathfrak{N}$), then $\mathfrak{M} \models d_0 \cdot d_1 = c$ iff $\mathfrak{N} \models d_0 \cdot d_1 = c$.

If $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e, $d \leq e$ and $\theta(x)$ is a Δ_0 formula, it follows that $\mathfrak{M} \models \theta(d)$ iff $\mathfrak{N} \models \theta(d)$.

Theorem 11. *$PA + \mathrm{Con}^*(PA) \not\vdash \mathrm{Con}(PA)$.*

Proof: Let $\mathfrak{M}$ be a countable non-standard model of PA + F_{ε_0} *is total.* Let M be the domain of $\mathfrak{M}$ and $a \in M$ be non-standard. Also let $e = F^{\mathfrak{M}}_{\varepsilon_0}(a)$. As a result of the assumption that F_{ε_0} is total in $\mathfrak{M}$, $\mathfrak{M} \models \mathrm{Con}(\mathrm{PA}\restriction_a)$. By a result of Solovay, there exists a countable model $\mathfrak{N}$ of PA such that:

(i) $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e (in the sense of Definition 10).

(ii) $\mathfrak{N}$ thinks that $\mathrm{PA}\restriction_a$ is consistent.

(iii) $\mathfrak{N}$ thinks that $\mathrm{PA}\restriction_{a+1}$ is inconsistent. In fact in $\mathfrak{N}$ there is a proof of $0 = 1$ from $\mathrm{PA}\restriction_{a+1}$ whose Gödel number is less than 2^{2^e} (as computed in $\mathfrak{N}$).

Actually, to be able to apply Solovay's theorem we have to ensure that e is much larger than a, i.e., $E^k(a) < e$ for every standard number k. It is a standard fact (provable in PA) that $E(x) \leq F_3(x)$ holds for all sufficiently large x. In particular this holds for all non-standard elements s of $\mathfrak{M}$ and hence

$$E^k(s) \leq F_3^k(s) \leq F_3^s(s) \leq F_4(s) < F_{\varepsilon_0}(s),$$

so that $E^k(a) < e$ holds for all standard k, leading to e being much larger than a.

We now distinguish two cases.

Case 1: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\uparrow$. Then also $\mathfrak{N} \models F_{\varepsilon_0}(d)\uparrow$ for all $d > a$. Hence, in light of (ii), $\mathfrak{N} \models \mathrm{Con}^*(\mathrm{PA})$. As (iii) yields $\mathfrak{N} \models \neg\mathrm{Con}(\mathrm{PA})$, we have

$$\mathfrak{N} \quad \models \quad \mathrm{PA} + \mathrm{Con}^*(\mathrm{PA}) + \neg\mathrm{Con}(\mathrm{PA}). \tag{3}$$

Case 2: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\downarrow$. We then also have $e = F^{\mathfrak{N}}_{\varepsilon_0}(a)$, for $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e and the formula '$F_{\varepsilon_0}(x) = y$' is Δ_0. Let $c := F^{\mathfrak{N}}_{\varepsilon_0}(a+1)$. By Theorem 9, for every standard n there is an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction. Moreover, it follows from the properties of $\mathfrak{N}$ and the fact that $2^{2^e} < \mathfrak{I}$, that

1. $\mathfrak{I}$ thinks that $\mathrm{PA}\restriction_a$ is consistent.

2. $\mathfrak{I}$ thinks that $\mathrm{PA}\restriction_{a+1}$ is inconsistent.

3. $\mathfrak{I}$ thinks that $F_{\varepsilon_0}(a+1)$ is not defined.

Consequently, $\mathfrak{I} \models \mathrm{Con}^*(\mathrm{PA}) + \neg\mathrm{Con}(\mathrm{PA}) + \Pi_{n+1}$-induction. Since n was arbitrary, this shows that $\mathrm{PA} + \mathrm{Con}^*(\mathrm{PA}) + \neg\mathrm{Con}(\mathrm{PA})$ is consistent. $\square$

Proposition 6 and Theorem 11 can be extended to theories $\mathbf{T} = \mathrm{PA} + \psi$ where ψ is any true Π^0_1 statement.

Theorem 12. *Let* $\mathbf{T} = PA + \psi$ *where* ψ *is a* Π_1 *statement such that* $T +$ 'F_{ε_0} is total' *is a consistent theory. Let* $\mathbf{T}\restriction_k$ *to be the theory* $PA\restriction_k + \psi$ *and* $\mathrm{Con}^*(\mathbf{T}) := \forall x \mathrm{Con}(\mathbf{T}\restriction_{F^{-1}_{\varepsilon_0}(x)})$. *Then the strength of* $\mathbf{T} + \mathrm{Con}^*(T)$ *is strictly between* $\mathbf{T}$ *and* $\mathbf{T} + \mathrm{Con}(\mathbf{T})$, *i.e.*

(i) $\mathbf{T} \nvdash \mathrm{Con}^*(\mathbf{T})$.

(ii) $\mathbf{T} + \mathrm{Con}^*(\mathbf{T}) \nvdash \mathrm{Con}(\mathbf{T})$.

(iii) $\mathbf{T} + \mathrm{Con}(\mathbf{T}) \vdash \mathrm{Con}^*(\mathbf{T})$.

Proof: For (i) the same proof as in Proposition 6 works with PA replaced by $\mathbf{T}$. (iii) is obvious. For (ii) note that Solovay's Theorem also works for $\mathbf{T}$ so that the proof of case 1 of Theorem 11 can be copied. To deal with case

2, observe that $\mathfrak{I} \models \psi$ since ψ is Π_1, $\mathfrak{N} \models \psi$ and $\mathfrak{I}$ is an initial segment of $\mathfrak{N}$. $\square$

The methods of Theorem 11 can also be used to produce two "natural" slow growing functions f and g such that the theories $\mathrm{PA} + \mathrm{Con}_f(\mathrm{PA})$ and $\mathrm{PA} + \mathrm{Con}_g(\mathrm{PA})$ are mutually non-interpretable in each other.

Definition 13. The even and odd parts of F_{ε_0} are defined as follows:

$$F^{even}_{\varepsilon_0}(2n) = F_{\varepsilon_0}(2n), \qquad F^{even}_{\varepsilon_0}(2n+1) = F_{\varepsilon_0}(2n) + 1\,,$$
$$F^{odd}_{\varepsilon_0}(2n+1) = F_{\varepsilon_0}(2n+1), \qquad F^{odd}_{\varepsilon_0}(2n+2) = F_{\varepsilon_0}(2n+1) + 1, \quad F^{odd}_{\varepsilon_0}(0) = 1,$$

$$\begin{array}{rcl} f(n) & = & \max(\{k \le n \mid \exists y \le n\, F^{even}_{\varepsilon_0}(k) = y\} \cup \{0\}) \\ g(n) & = & \max(\{k \le n \mid \exists y \le n\, F^{odd}_{\varepsilon_0}(k) = y\} \cup \{0\}). \end{array}$$

The graphs of f and g are Δ_0 and both functions are provably recursive functions of PA.

By a small variation on the proof of Theorem 11 we get:

Theorem 14. *(i)* $PA + \mathrm{Con}_f(PA) \nvdash \mathrm{Con}_g(PA)$.

(ii) $PA + \mathrm{Con}_g(PA) \nvdash \mathrm{Con}_f(PA)$.

A natural Orey sentence

A sentence φ is called an *Orey sentence* for PA if both $\mathrm{PA} + \varphi \lhd \mathrm{PA}$ and $\mathrm{PA} + \neg\varphi \lhd \mathrm{PA}$ hold. An natural Orey sentence for ZFC is CH.

Corollary 15. *The sentence* $\exists x\, (F_{\varepsilon_0}(x)\!\uparrow\, \wedge \forall y < x\, F_{\varepsilon_0}(y)\!\downarrow\, \wedge\, x\ \texttt{is even})$ *is an Orey sentence.*

Proof: Let ψ be the above sentence. It suffices to show that $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA}\restriction_k + \psi)$ and $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA}\restriction_k + \neg\psi)$ hold for all k. Fix $k > 0$.

First we show that $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA}\restriction_k + \psi)$. Note that PA proves the consistency of $\mathrm{PA}\restriction_k + \exists x F_{\varepsilon_0}(x)\uparrow$. Arguing in PA we thus find a non-standard model $\mathfrak{N}$ such that

$$\mathfrak{N} \models \mathrm{PA}\restriction_k + \exists x F_{\varepsilon_0}(x)\uparrow .$$

In particular there exists a least $a \in |\mathfrak{N}|$ in the sense of $\mathfrak{N}$ such that $\mathfrak{N} \models F_{\varepsilon_0}(a)\uparrow$. If $\mathfrak{N}$ thinks that a is even, then $\mathfrak{N} \models \psi$, which entails that $\mathrm{Con}(\mathrm{PA}\restriction_k$

$+\psi$). If $\mathfrak{N}$ thinks that a is odd, we define a cut $\mathfrak{I}$ such that $\mathfrak{I} \models \mathrm{PA}\restriction_k$ and $F^{\mathfrak{N}}_{\varepsilon_0}(a-2) < \mathfrak{I} < F^{\mathfrak{N}}_{\varepsilon_0}(a-1)$, applying Theorem 9. Then $\mathfrak{I} \models \psi$ which also entails $\mathrm{Con}(\mathrm{PA}\restriction_k + \psi)$.

Next we show that $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA}\restriction_k + \neg\psi)$. As PA proves $\mathrm{Con}(\mathrm{PA}\restriction_k)$, we can argue in PA and assume that we have a model $\mathfrak{M} \models \mathrm{PA}\restriction_k$. If $\mathfrak{M} \models \forall x F_{\varepsilon_0}(x)\downarrow$ then $\mathfrak{M} \models \neg\psi$, and $\mathrm{Con}(\mathrm{PA}\restriction_k + \neg\psi)$ follows. Otherwise there is a least a in the sense of $\mathfrak{M}$ such that $F^{\mathfrak{M}}_{\varepsilon_0}(a)\uparrow$. If $\mathfrak{M}$ thinks that a is odd we have $\mathfrak{M} \models \neg\psi$, too. If $\mathfrak{M}$ thinks that a is even we introduce a cut $F^{\mathfrak{M}}_{\varepsilon_0}(a-2) < \mathfrak{I}' < F^{\mathfrak{M}}_{\varepsilon_0}(a-1)$ such that $\mathfrak{I}' \models \mathrm{PA}\restriction_k$. Since $\mathfrak{I}' \models F_{\varepsilon_0}(a-1)\uparrow$ we have $\mathfrak{I}' \models \neg\psi$, whence $\mathrm{Con}(\mathrm{PA}\restriction_k + \neg\psi)$. $\square$

Some further remarks about slow consistency

1. It is quite natural to consider another version of slow consistency where the function $f : \mathbb{N} \to \mathbb{N}$, rather than acting as a bound on the fragments of PA, restricts the lengths of proofs. Let $\perp$ be a Gödel number of the canonical inconsistency and let $\mathrm{Proof}_{\mathrm{PA}}(y, z)$ be the primitive recursive predicate expressing the concept that *"y is the Gödel number of a proof in PA of a formula with Gödel number z"*.

$$\mathrm{Conl}_f(\mathrm{PA}) \quad := \quad \forall x \, \forall y < f(x) \, \neg \mathrm{Proof}_{\mathrm{PA}}(y, \perp) \tag{4}$$

Let $\mathrm{Con}^{\#}(\mathrm{PA})$ be the statement $\mathrm{Conl}_{F^{-1}_{\varepsilon_0}}(\mathrm{PA})$.

Note that $\mathrm{Con}^{\#}(\mathrm{PA})$ is equivalent to the following formula:

$$\forall u \, [F_{\varepsilon_0}(u)\downarrow \to \ \forall y < u \, \neg \mathrm{Proof}_{\mathrm{PA}}(y, \perp)].$$

As it turns out, by contrast with $\mathrm{Con}^*(\mathrm{PA})$, $\mathrm{Con}^{\#}(\mathrm{PA})$ is not very interesting.

Proposition 16. *$PA \vdash \mathrm{Con}^{\#}(PA)$.*

2. Iterating slow consistency: What happens if we consider the sequence of theories

$\mathrm{PA} \lhd \mathrm{PA} + \mathrm{Con}^*(\mathrm{PA}) \lhd \mathrm{PA} + \mathrm{Con}^*(\mathrm{PA} + \mathrm{Con}^*(\mathrm{PA})) \lhd \cdots$?

Proposition 17. *Setting $T_0 = PA$ and $T_{n+1} = T_n + \mathrm{Con}^*(T_n)$ we have*

$$T_m \lhd PA + \mathrm{Con}(PA)$$

for all m.

3. Con* can be weakened further by considering even slower-growing functions. Every ordinal analysis of a theory T gives rise to a hierarchy of functions $(F_\alpha)_{\alpha \leq \tau}$ such that:

i. Each F_α, $\alpha < \tau$ is provably recursive in T.
ii. Every provably recursive function of T is dominated by some F_α, $\alpha < \tau$.
iii. F_τ is not provably recursive in T and dominates any function which is provably recursive in T.

Now if τ is greater than ε_0 we will have a strict chain:

$$\mathrm{PA} \lhd \mathrm{PA} + \mathrm{Con}_{F_\tau^{-1}}(\mathrm{PA}) \lhd \mathrm{PA} + \mathrm{Con}^*(\mathrm{PA}).$$

This yields an infinite descending chain of natural theories between PA and PA + Con(PA).

Global Weakenings of the Consistency Operator

In my wish to show that $T+\mathrm{Con}(T)$ is the least "natural" strengthening of T, I considered global versions of the consistency operator. Let Φ be a computable function from sentences of arithmetic to Π_1 sentences of arithmetic. Then Φ is a *proof-theoretic jump operator* iff:

1. For any sentence φ consistent with PA, PA $+ \varphi$ does not prove $\Phi(\varphi)$.
2. If φ, ψ are provably equivalent in PA then $\Phi(\varphi), \Phi(\psi)$ are provably equivalent in PA.

Of course the standard example of such an operator is $\Phi(\varphi) = \mathrm{Con}(\mathrm{PA} + \varphi)$. My hope is that one can show that this is the "least" such operator in some sense.

Conjecture. If Φ is a proof-theoretic jump operator then PA $+ \Phi(\varphi)$ proves $\mathrm{Con}(\mathrm{PA} + \varphi)$ for all PA-consistent φ.

Shavrukov and Visser have refuted a strengthening of this:

Theorem 18. *(Shavrukov-Visser) There is a proof-theoretic jump operator Φ such that for all Π_1 sentences φ consistent with PA, PA $+\varphi$ does not prove $\Phi(\varphi)$ and PA $+ \varphi + \Phi(\varphi)$ does not prove* $\mathrm{Con}(PA + \varphi)$.

They also show the same thing with Π_1 replaced by Π_n for any n and with no restriction on φ but allowing the complexity of $\Phi(\varphi)$ to depend on that of φ.

5.-6. Vorlesungen

Topic 3: RE Degrees in Set Theory

Recall that the aim of slow consistency was to find a "natural" example of a statement between PA and PA + Con(PA) in strength. This is analagous to the problem of finding "natural" degrees between 0 and $0'$, the degree of the halting set, in computability theory.

Another context in which this problem of "natural intermediate degrees" can be posed is set theory. The setting of classical computation theory is ω. To motivate our generalisation in set theory it is convenient to think of this context as L_ω and for A, B subsets of L_ω define:

A is Turing reducible to B iff A is Δ_1 definable over the structure $(L_\omega, \in, B)$.

Now assume $V = L$ and let κ be an inaccessible cardinal. For A, B subsets of L_κ say that A is *κ-reducible to B* iff:

$$A \text{ is } \Delta_1 \text{ definable over the structure } (L_\kappa, \in, B).$$

Then κ-reducibility is transitive and we get an upper semilattice of κ-degrees. Moreover we can define the *κ-RE* sets to be the subsets of L_κ which are Σ_1 definable over $(L_\kappa, \in)$ and obtain an analogue of the Turing jump via:

A', the *κ-jump of A*, is the canonical universal Σ_1 predicate for the structure $(L_\kappa, \in, A)$, i.e.

$$\{(\varphi, x) \mid (L_\kappa, \in, A) \vDash \varphi(x) \text{ where } \varphi \text{ is } \Sigma_1 \text{ and } x \in L_\kappa\}.$$

It is straightforward to verify that A' has the largest κ-degree of any κ-RE set and the κ-degree of A' is greater than the κ-degree of A.

In the 1970s and 1980s a lot of work was done exploring this generalisation of computability theory (replacing κ by an arbitrary "admissible" ordinal). What I want to say here is that in the case of an inaccessible κ, we can exhibit "natural" examples of κ-RE degrees between 0 and $0'$.

Theorem 19. *Let $\delta < \kappa$ be an infinite regular cardinal and S_δ the set of ordinals less than κ of cofinality δ. Then S_δ is κ-RE and the κ-degree of S_δ is strictly between 0 and $0'$. Moreover the κ-degree of S'_δ equals $0'$.*

For simplicity we'll assume that δ is ω. I should mention that the proof below has nothing to do with the inaccessibility of κ, we only need that κ is a limit cardinal in L (or with a little extra effort, even weaker: L_κ thinks that there is no largest cardinal).

The main thing to show is the following:

Lemma 20. *Suppose that α is less than κ. Then $(L_{\alpha^{++}}, \in, S_\omega \cap \alpha^{++})$ is a Σ_1 elementary submodel of $(L_\kappa, \in, S_\omega)$.*

This gives us that the κ-jump S'_ω of S_ω is κ-reducible to $0'$: Recall that S'_ω is the set of pairs (φ, x) where φ is Σ_1, x belongs to L_κ and $(L_\kappa, \in, S_\omega) \vDash \varphi(x)$. But by the Lemma we have:

$(L_\kappa, \in, S_\omega) \vDash \varphi(x)$ iff
$(L_{\alpha^{++}}, \in, S_\omega \cap \alpha^{++}) \vDash \varphi(x)$ when x belongs to $L_{\alpha^{++}}$.

And the set of cardinals less than κ is Π_1 definable over L_κ and therefore κ-reducible to $0'$. It follows that the function that chooses for each $x \in L_\kappa$ the least α such that x belongs to $L_{\alpha^{++}}$ is also κ-reducible to $0'$. So S'_ω is also κ-reducible to $0'$.

Since the κ-jump of S_ω is κ-reducible to $0'$ it follows that the κ-degree of S_ω is less than $0'$. Later we'll also verify that S_ω is not κ-reducible to 0.

Proof of Lemma 20. Suppose that φ is Σ_1, $x \in L_{\alpha^{++}}$ and $(L_\kappa, \in, S_\omega) \vDash \varphi(x)$. We must show that $(L_{\alpha^{++}}, \in, S_\omega \cap \alpha^{++}) \vDash \varphi(x)$. Choose an ordinal $\delta < \alpha^{++}$ such that x belongs to L_δ. Now let M be an elementary submodel of $(L_\kappa, \in, S_\omega)$ such that:

i. L_δ is a subset of M.
ii. M is ω-closed: Any countable subset of M is an element of M.
iii. M has cardinality α^+.

It is easy to find such an M: Start with L_δ, closing under Skolem functions and ω-sequences, and repeat this ω_1 times. Note that the cardinality will stay less than α^{++} as α^{++} is the successor of an uncountable regular cardinal.

Let $\pi : M \simeq \bar{M}$ be the transitive collapse of M. As L_δ is a subset of M and x belongs to L_δ we have that $\pi(x) = x$.

Now we have:

$(L_\kappa, \in, S_\omega) \models \varphi(x)$, so
$M \models \varphi(x)$, so
$\bar{M} \models \varphi(x)$.

Now $\bar{M}$ looks like $(L_{\bar{\kappa}}, \in, \bar{S})$ for some ordinal $\bar{\kappa}$, where $\bar{S}$ is the "image" of S_ω under π, i.e., the set of ordinals less than $\bar{\kappa}$ which have cofinality ω in $L_{\bar{\kappa}}$. But since M is ω-closed, so is $L_{\bar{\kappa}}$ and therefore:

$\bar{S} = S_\omega \cap \bar{\kappa}$.

So $\bar{M} = (L_{\bar{\kappa}}, \in, S_\omega \cap \bar{\kappa}) \models \varphi(x)$ and therefore since φ is Σ_1, $(L_{\alpha^{++}}, \in, S_\omega \cap \alpha^{++}) \models \varphi(x)$, as desired. $\square$

Now we give the promised argument that S_ω is not κ-reducible to 0. Otherwise, the complement of S_ω is definable over $(L_\kappa, \in)$ by some Σ_1 formula with some parameter p. Choose α so that p belongs to L_{α^+}. Then we have $(L_\kappa, \in) \models \varphi(\alpha^{++}, p)$. But now let M be an elementary submodel of L_κ such that $M \cap \alpha^{++}$ is an ordinal $\delta < \alpha^{++}$ of cofinality ω. (This is possible by taking union of an ω-chain of elementary submodels containing L_{α^+} which have increasing intersections with α^{++}.) Let $\pi : M \simeq \bar{M}$ be the transitive collapse of M. Then $\pi(\alpha^{++})$ is an ordinal of cofinality ω, but as $\bar{M} \models \varphi(\pi(\alpha^{++}), p)$ it follows that $L_\kappa \models \varphi(\pi(\alpha^{++}), p)$, contradicting the fact that φ and p define the complement of S_ω.

So S_ω has κ-degree strictly between 0 and $0'$.

The same argument shows that S_δ has κ-degree strictly between 0 and $0'$ for any infinite regular $\delta < \kappa$. A further argument shows:

Theorem 21. *For δ_0, δ_1 infinite regular cardinals less than κ, the sets $S_{\delta_0}, S_{\delta_1}$ are incomparable under κ-reducibility.*

Open questions remain: Can one get a "natural" example of a κ-degree which is between 0 and $0'$ but not "low" (i.e., whose κ-jump is not computable from $0'$)? Do the distinct $S_{\delta_0}, S_{\delta_1}$ form a minimal pair?

Vaught was probably the first to start looking at the isomorphism relation on the countable models of a theory in terms of descriptive set theory. His point was that countable models (for a countable language) can be coded by reals and then the equivalence relation

xEy iff M_x is isomorphic to M_y

(where M_x is the model coded by x) is an analytic (i.e., a boldface Σ^1_1) equivalence relation. It is natural to ask: Do all analytic equivalence relations on the reals look like this, and if not, how do these "logic equivalence relaions" sit inside the class of analytic equivalence relations as a whole?

First I make a few basic observations. We don't have to restrict ourselves to first-order theories when looking at isomorphism relations as above. Indeed, let φ be any sentence of the infinitary logic $L_{\omega_1\omega}$, which allows countably infinite conjunctions and disjunctions (in addition to finite strings of quantifiers). Then

$\{x \mid M_x \vDash \varphi\}$

is a Borel set and we again get an analytic equivalence relation if we restrict isomorphism to the models of φ. Actually, a theorem of Lopez-Escobar says that conversely, any Borel set B which is *invariant*, i.e. satisfies

$x \in B$, M_x isomorphic to M_y implies $y \in B$

is of the above form for some φ. So when talking about isomorphism relations on countable models of a theory we really should be considering theories which are described by a sentence of $L_{\omega_1\omega}$.

Secondly, note that these isomorphism relations are really quite special, as revealed by the following.

Theorem 22. *(Scott) For any real x, $\{y \mid M_x$ is isomorphic to $M_y\}$ is a Borel set.*

Now for an arbitrary analytic equivalence relation E on the reals this may fail. For example, let A be any analytic set of reals that is not Borel and define

xEy iff $x, y \in A$ or $x = y$.

This is an analytic equivalence relation with the non-Borel set A as one of its equivalence classes. It follows that this E is not "reducible" to any isomorphism relation in any reasonable sense.

7.Vorlesung

Borel reducibility

What is a reasonable sense of "reducibility" between analytic equivalence relations? If E, F are equivalence relations on Polish spaces X, Y then we write

$$E \leq_B F$$

iff there is a Borel reduction from E to F, i.e., a Borel function $f : X \to Y$ so that for any x_0, x_1 in X:

$$x_0 E x_1 \text{ iff } f(x_0) F f(x_1).$$

We write $\sim_B$ for ($\leq_B$ and $\geq_B$) and $<_B$ for ($\leq_B$ and $\not\geq_B$).

There are $\leq_B$-complete isomorphism relations, to which all isomorphism relations are Borel-reducible (examples below). A $\leq_B$-complete isomorphism relation is necessarily analytic and not Borel. I next describe a $\leq_B$-cofinal hierarchy of Borel isomorphism relations, together with some examples that occur at particular levels of this hierarchy. The hierarchy looks like this:

$0 < 1 < \cdots < \omega < \mathrm{id}_R < E_0 < E_\infty < F_2 < F_3 < \cdots < F_\alpha < \cdots (\alpha < \omega_1)$

1. Borel-equivalent to ω.

Finite linear orderings

2. Borel-equivalent to id_R = equality on the reals

Orderings of type ω with a unary predicate

Smooth = Borel-reducible to id_R.

3. E_0 = equality mod finite on subsets of ω

Subgroups of $(Q, +)$

4. $E_\infty = \leq_B$-largest *countable* Borel equivalence relation

Countable = has countably many equivalence classes.

Fact: Any countable Borel equivalence relation is Borel bireducible to an isomorphism relation.

Locally-finite, connected graphs
Finitely-generated groups
Fields of finite transcendence degree over Q

5. F_α

xF_2y iff $\{x_n \mid n \in \omega\} = \{y_n \mid n \in \omega\}$

xF_3y iff $\{\{(x_m)_n \mid n \in \omega\} \mid m \in \omega\}$ = same for y

etc.

Each Borel isomorphism relation is Borel-reducible to some F_α

Equivalent to F_2:

locally-finite graphs
Archimedean totally-ordered Abelian groups with a distinguished positive element

6. Beyond Borel

$\leq_B$-complete isomorphism relations: graphs, trees, fields, groups, linear orderings, Boolean algebras

Abelian p-groups: Invariants are elements of $2^{<\omega_1}$ (Ulm invariants). Not complete.

Torsion-free Abelian groups: Not known to be complete.

7. Beyond isomorphism relations

Hjorth: An orbit equivalence relation is reducible to an isomorphism relation iff it is not "turbulent".

Non-turbulent = invariants are given by countable structures

Examples of turbulent actions:

Conjugacy on the homeomorphism group of the unit square
Conjugacy of ergodic, measure-preserving transformations
Unitary equivalence of unitary operators
Biholomorphic equivalence of 2-dimensional complex manifolds

8. Beyond orbit equivalence relations

$E_1 : xE_1y$ iff $\{x_n \mid n \in \omega\}$ almost equals $\{y_n \mid n \in \omega\}$

Composant equivalence relation for certain indecomposable continua

8.Vorlesung

Bi-embeddability relations

As indicated above, the isomorphism relations are very far from capturing arbitrary analytic equivalence relations up to Borel bireducibility. We'll show now however that a related model-theoretic notion, that of bi-embeddability, is sufficient to do this.

A *quasi-order* is a reflexive, transitive relation. If R is a quasi-order then the *equivalence relation derived from* R is given by xEy iff (xRy and yRx). An *embeddability relation* is the restriction of the quasi-order of embeddability to the countable models of a sentence of $\mathcal{L}_{\omega_1\omega}$ (for some countable language $\mathcal{L}$).

First observe the following.

Theorem 23. *There is a $\leq_B$-complete analytic quasi-order.*

Proof. Let W_0 be an analytic subset of $(2^\omega)^3$ which is universal for analytic subsets of $(2^\omega)^2$, i.e., any analytic subset of $(2^\omega)^2$ is of the form $\{(y,z) \mid (x,y,z) \in W_0\}$ for some x. Define W by: $(x_1,y_1)W(x_2,y_2)$ iff $(x_1 = x_2 \wedge \exists z_1,\ldots,z_n(z_1 = y_1,\ z_n = y_2$ and $(x_1,z_i,z_{i+1}) \in W_0$ for $1 \leq i < n)$. Then W

is a complete analytic quasi-order: Clearly it is an analytic quasi-order, as analytic relations are closed under existential quantification over reals. If R is any analytic quasi-order on 2^ω with W_0-code x, then

$$y_1 R y_2 \leftrightarrow (x, y_1) W (x, y_2),$$

and therefore the map $y \mapsto (x, y)$ reduces R to W. □

Theorem 24. *Let E be an analytic equivalence relation on a Polish space X. Then E is $\leq_B$-complete as an analytic equivalence relation iff E is the equivalence relation $\equiv_R$ derived from a $\leq_B$-complete analytic quasi-order R on X.*

Proof. Suppose that R is a complete analytic quasi-order. If F is an analytic equivalence relation on a Polish space, then F is in particular a quasi-order and therefore is Borel-reducible to R. But then the same reduction shows that F also Borel-reduces to $\equiv_R$, which is therefore a complete analytic equivalence relation.

Conversely, suppose that E is a complete analytic equivalence relation on X and let R_0 be a complete analytic quasi-order on 2^ω. Let $f : 2^\omega \to X$ be a Borel reduction of $\equiv_{R_0}$, the equivalence relation derived from R_0, to E. Define:

$$xRy \leftrightarrow xEy \vee \exists a \exists b (xEf(a) \wedge yEf(b) \wedge aR_0 b).$$

Then R is analytic and contains E. Let X_0 be $\{x \mid \exists a(xEf(a))\}$ and $X_1 = X \setminus X_0$. Then f is a reduction of R_0 to R restricted to X_0 and the latter is a quasi-order whose derived equivalence relation equals E restricted to X_0. And R restricted to X_1 equals E. It follows that R is an analytic quasi-order whose derived equivalence relation is E and as f is a reduction of R_0 to R, it follows that R is a complete analytic quasi-order. □

Theorem 25. *There is an embeddability relation which is complete as an analytic quasi-order (and therefore there is a bi-embeddability relation which is complete as an analytic equivalence relation).*

Proof. First we introduce a particular complete analytic quasi-order $\leq_{max}$, and then use it to show that a certain embeddability relation is also complete.

If s, t are finite sequences from ω of the same length, then we write $s \leq t$ iff $s(i) \leq t(i)$ for all $i < |s|$ and $s + t$ for the sequence of length $|s|$ whose value at i is $s(i)+t(i)$. For any set X, a *tree* on X is a subset of $X^{<\omega}$ closed under restriction. If T is a tree on $X\times\omega$ then we view elements of T not as sequences of pairs (x,n) from $X\times\omega$ but rather as pairs (u,s) of sequences $u\in X^{<\omega}, s\in\omega^{<\omega}$ of the same length. We say that T is *normal* iff whenever (u,s) belongs to T and $s\leq t$ then (u,t) belongs to T. For $s\in\omega^{<\omega}$ set $T(s)=\{u\in X^{<\omega}\mid |u|=|s|\wedge(u,s)\in T\}$. Thus for normal T we have $s\leq t\rightarrow T(s)\subseteq T(t)$.

A function $f:\omega^{<\omega}\rightarrow\omega^{<\omega}$ is *Lipschitz* iff f preserves both length and extension (i.e. $|s|=|f(s)|$ for each s and $s\subseteq t\rightarrow f(s)\subseteq f(t)$ for each s,t).

Definition 26. *Let $\mathcal{T}$ be the space of normal trees on $2\times\omega$, with its natural Polish topology. Define $\leq_{max}$ on $\mathcal{T}$ by:*

$$S\leq_{max}T\leftrightarrow\exists\ \textit{Lipschitz}\ f:\omega^{<\omega}\rightarrow\omega^{<\omega}\ \forall s\in\omega^{<\omega}\ S(s)\subseteq T(f(s)).$$

This is a strong way of saying that the projection of S is included in the projection of T.

9.-10.Vorlesungen

$\leq_{max}$ is an analytic quasi-order on $\mathcal{T}$. To prove that it is complete, we use the following "normal form" result for analytic quasi-orders on 2^ω.

Lemma 27. *Let R be an analytic quasi-order on 2^ω. Then there is a tree S on $2\times2\times\omega$ such that:*
(i) R is the projection of S, i.e., xRy iff for some z, $(x|n,y|n,z|n)\in S$ for all n.
(ii) S is normal, i.e., if (u,v,s) belongs to S and $s\leq t$ then (u,v,t) belongs to S.
(iii) If $u\in2^{<\omega}$ and $s\in\omega^{<\omega}$ have the same length, then (u,u,s) belongs to S.
(iv) If (u,v,s) and (v,w,t) belong to S then so does $(u,w,s+t)$.
(v) If $u,v\in2^{<\omega}$ have the same length then $(u,v,0^{|u|})\in S$ implies $u=v$.

Proof. Start with any tree T_0 on $2\times2\times\omega$ with R the projection of T_0. If we set $T_1=\{(u,v,t)\mid\exists s\leq t(u,v,s)\in T_0\}$ then T_1 is normal and we have by König's lemma that R is also the projection of T_1. Also, if we let T_2 be $T_1\cup\{(u,u,s)\mid|u|=|s|\}$ then T_2 satisfies (i),(ii) and (iii).

Now define S by: $(\emptyset, \emptyset, \emptyset) \in S$ and for $k, n \in \omega$, $u, v \in 2^k$, $s \in \omega^k$, $i, j \in 2$: $(u * i, v * j, n * s) \in S$ iff $\exists u_0, u_1, \ldots, u_n \in 2^k(u_0 = u \wedge u_n = v \wedge \forall l < n(u_l, u_{l+1}, s) \in T_2)$. (So if $n = 0$, $(u * i, v * j, 0 * s) \in S$ iff $u = v$.)

Then S has properties (i-iv): Clearly it is a tree. To check (i), note first that if (x, y, a) is a branch through T_2 then $(x, y, 1 * a)$ is a branch through S. So R is a subset of the projection of S. Conversely, suppose that $(x, y, n * a)$ is a branch through S. If $n = 0$, $x = y$ and $(x, y) \in R$. If $n > 0$ we get for each k sequences $(u_i^k)_{i \leq n}$ in 2^k with $u_0^k = x|k$, $u_n^k = y|k$ and for $i < n$, $(u_i^k, u_{i+1}^k, a|k) \in T_2$. By the compactness of 2^ω, we can find a subsequence (k_l) and for $i \leq n$ elements z_i of 2^ω such that $u_i^{k_l} \to z_i$, as $l \to \infty$. But then for $i < n$, (z_i, z_{i+1}, a) is a branch through T_2, hence $z_i R z_{i+1}$. As $z_0 = x$ and $z_n = y$, by transitivity we get xRy, as desired.

To check (ii), let $(u, v, s) \in S$ and $t \geq s$. The case of $(\emptyset, \emptyset, \emptyset)$ is trivial. So suppose $u = u' * i$, $v = v' * j$, $s = n * s'$ and $t = m * t'$, with $n \leq m$ and $s' \leq t'$. As T_2 is normal we also have $(u, v, n * t') \in S$, with the same witnesses $(u_i)_{i \leq n}$. Also, using property (iii) of T_2 we can repeat the witness u_0 $(m - n)$ times to get witneses for $(u, v, m * t') \in S$, as desired.

(iii) follows from (ii) and the fact that if $|u| = |s|$ and $s(0) = 0$ then $(u, u, s) \in S$.

To check (iv), let $u = u' * i$, $v = v' * j$, $w = w' * k$, $s = n * s'$ and $t = m * t'$ satisfy $(u, v, s) \in S$ and $(v, w, t) \in S$. By (ii) we also have $(u, v, n * (s' + t')) \in S$ and $(v, w, m * (s' + t')) \in S$, as witnessed by say $(u_i)_{i \leq n}$, $(v_j)_{j \leq m}$. But then $(u_i)_{i < n} * (v_j)_{j \leq m}$ is a witness that $(u, w, (n + m) * (s' + t')) \in S$, as desired.

Finally, to obtain (v) simply modify S by discarding elements of the form $(u, v, 0^{|u|})$ when u, v are of the same length and different. This preserves properties (i-iv). $\Box$ (Lemma 27)

Now we show that any analytic quasi-order R on 2^ω is Borel-reducible to $\leq_{max}$. Let S be the tree associated to R by the lemma and define $f : 2^\omega \to \mathcal{T}$ by

$$f(x) = S^x = \{(u, s) \in (2 \times \omega)^{<\omega} \mid (u, x \restriction |u|, s) \in S\}.$$

The tree S^x is normal as S is. We check that f is the desired Borel reduction. Suppose first that $S^x \leq_{max} S^y$, witnessed by the Lipschitz map $\varphi : \omega^{<\omega} \to \omega^{<\omega}$. Then the sequences $\varphi(0^k)$, $k \in \omega$, extend each other and hence build some $a \in \omega^\omega$. By property (iii), for all k, $(x|k, 0^k) \in S^x$, hence $(x|k, \varphi(0^k)) \in S^y$. So (x, y, a) is a branch through S and by (i), xRy.

Conversely, suppose xRy and let a be such that (x, y, a) is a branch through S. Define $\varphi : \omega^{<\omega} \to \omega^{<\omega}$ by $\varphi(s) = s + (a \restriction |s|)$. The map φ is clearly Lipschitz. We must show that for each s, $u \in S^x(s)$ implies $u \in S^y(\varphi(s))$. Suppose $u \in S^x(s)$ and $|u| = k$. Then we have $(u, x|k, s) \in S$, and as (x, y, a) is a branch through S we also have $(x|k, y|k, a|k) \in S$. Hence by property (iv) of S, $(u, y|k, s + a|k) = (u, y|k, \varphi(s)) \in S$ and so $(u, \varphi(s)) \in S^y$. Thus φ witnesses $S^x \leq_{max} S^y$, as desired.

Also note that this reduction of R to $\leq_{max}$ is injective, as using properties (iii,v) of Lemma 27 we have that if $x|k \neq y|k$ then $(x|k, 0^k) \in S^x$ but $(x|k, 0^k) \notin S^y$.

Finally we Borel-reduce $\leq_{max}$ to a particular embeddability relation, namely, the embeddability relation on (countable) *combinatorial trees*, i.e., symmetric, irreflexive, connected, acyclic binary relations. Fix some injection θ of $2^{<\omega}$ into ω such that $|s| \leq |t|$ implies $\theta(s) \leq \theta(t)$. For each $T \in \mathcal{T}$ we describe the combinatorial tree G_T.

First we add, for each $s \in \omega^{<\omega} \setminus \{\emptyset\}$ another vertex s^* and put edges between s^* and s and between s^* and the predecessor s^- of s. This defines a combinatorial tree G_0. Then for each pair $(u, s) \in T$ we add vertices (u, s, x) where x is either 0^k or $0^{2\theta(u)+2} * 1 * 0^k$, for $k \in \omega$: Also, we link each (u, s, x) to (u, s, x') where x' is the predecessor of x (as a sequence) and link $(u, s, \emptyset)$ to s. This completely describes the combinatorial tree G_T.

We make some simple observations about G_T. First, one can compute the valence v_T (number of neighbours) of vertices in G_T: elements in $\omega^{<\omega}$ have valence ω, elements $(u, s, 0^{2\theta(u)+2})$, for $(u, s) \in T$, have valence 3, and all other vertices have valence 2. Next consider the distance d_T between vertices. The distance between vertices in $\omega^{<\omega}$ is even, and the distance between a vertex $(u, s, 0^{2\theta(u)+2})$ and points in $\omega^{<\omega}$ is odd and at least $2\theta(u) + 3$ (obtained at s).

11.-12.Vorlesungen

Suppose that $S \leq_{max} T$. Then there is in fact a 1-1 Lipschitz map $f : \omega^{<\omega} \to \omega^{<\omega}$ with $S(s) \subseteq T(f(s))$ for $s \in \omega^{<\omega}$. Define an embedding of G_S into G_T as follows: Send $s \in \omega^{<\omega}$ to $f(s)$ and s^* to $f(s)^*$. This defines an embedding of G_0 into itself. Next if $(u, s) \in S$ we have $(u, f(s)) \in T$ so we can send (u, s, x) to $(u, f(s), x)$. Thus G_S embeds into G_T.

Conversely, suppose that g is an embedding of G_S into G_T. Then we have $v_T(g(y)) \geq v_S(y)$ and $d_T(g(y), g(z)) = d_S(y, z)$ for all vertices y, z in the domain of G_S. Using the first of these facts, g must send elements in $\omega^{<\omega}$ to elements of $\omega^{<\omega}$, i.e., defines a map $f : \omega^{<\omega} \to \omega^{<\omega}$. We claim that f witnesses $S \leq_{max} T$. First we show $f(\emptyset) = \emptyset$: Consider $x = (\emptyset, \emptyset, 0^2)$. It is a vertex in G_S of valence 3 and d_S-distance 3 from $\emptyset$. So it must be sent to some vertex of valence at least 3 in G_T, witih d_T-distance 3 from $f(\emptyset)$. But there is only one possible such vertex, namely, $(\emptyset, \emptyset, 0^2)$, as points in $\omega^{<\omega}$ are at even distance from $f(\emptyset)$ and the other vertices of valence 3 are at a larger distance. This implies that $f(\emptyset) = \emptyset$. Second we show that f is Lipschitz. It suffices to show by induction on the length of s that $f(s)$ has the same length as s and extends $f(s^-)$ (when s is nonempty). The base case was done above. As $s*n$ has distance 2 from s in G_S, $f(s*n)$ must have distance 2 from $f(s)$ in G_T; it cannot be $f(s)^-$, which is $f(s^-)$ by induction. So it is $f(s)*k$ for some k, completing the induction. So f is Lipschitz. Finally, we show that if $(u, s) \in S$ then $(u, f(s)) \in T$. Consider the vertex $x = (u, s, 0^{2\theta(u)+2})$ in G_S. It must be sent by g to some vertex y in G_T of valence at least 3 and at distance $2\theta(u) + 3$ from $f(s)$. Again points in $\omega^{<\omega}$ are forbidden by parity, so $y = (v, t, 0^{2\theta(v)+2})$ for some $(v, t) \in T$. But as the path in G_S joining s to x does not contain s^-, the path in G_T joining $f(s)$ to y does not contain $f(s^-) = f(s)^-$, and so t must extend $f(s)$. But if it extends it strictly, we get $|v| = |t| > |f(s)| = |s| = |u|$ and $\theta(v) > \theta(u)$ so that the distance from y to $f(s)$ is too big. So $t = f(s)$ and $\theta(v) = \theta(u)$, hence $v = u$ and finally $(u, f(s)) \in T$, as desired. $\square$

For future use we need a modification of the above argument. The collection OCT of *ordered (countable) combinatorial trees* consists of those $G = \langle U_G, G, \leq_G \rangle$ such that $\langle U_G, G \rangle$ is a *combinatorial tree* (that is a connected and acyclic graph) and $\leq_G$ is a linear order of U_G.

Theorem 28. *The relation $\sqsubseteq_{OCT}$ of embeddability on OCT is complete for analytic quasi-orders.*

Proof. To each normal tree $T \in \mathcal{T}$ on $2 \times \omega$ associate the combinatorial tree G_T defined above.

Now define the order $\leq_T = \leq_{G_T}$ on the points in G_T in the following way: for $s, t \in \omega^{<\omega}$ put $s \preceq t$ if and only if $|s| < |t|$ or $|s| = |t|$ and $s \leq_{lex} t$ (the symbol $\prec$ will denote the strict part of $\preceq$). Also for nonempty $s, t \in \omega^{<\omega}$ put $s^* \preceq^* t^*$ iff $s \preceq t$. Now we order the points in G_T by:

- Points in $\omega^{<\omega}$ are less than points of either of the forms s^* or (u, s, x)
- Points of the form s^* are less than points of the form (u, t, x)
- Points in $\omega^{<\omega}$ are ordered by $\preceq$
- Points of the form s^* are ordered by $\preceq^*$
- $(u, s, x) \leq_T (v, t, y)$ iff

$$(s \prec t) \vee (s = t \wedge u \prec v) \vee (s = t \wedge u = v \wedge x \preceq y).$$

($\leq_T$ is a well-founded linear order.)

Now we show that the map $T \mapsto G_T$ is a reduction of $\leq_{max}$ to $\sqsubseteq_{OCT}$. Assume first that S, T are normal trees on $2 \times \omega$ such that $S \leq_{max} T$; then as above this can be witnessed by a Lipschitz $\leq_{lex}$-preserving function $f \colon \omega^{<\omega} \to \omega^{<\omega}$, that is by an f such that $s \preceq s' \iff f(s) \preceq f(s')$ for every $s, s' \in \omega^{<\omega}$ (in particular f is injective). Now embed G_S into G_T sending s to $f(s)$, s^* to $f(s)^*$, and (u, s, x) to $(u, f(s), x)$, and check that the order relations are preserved.

For the other direction, if $G_S \sqsubseteq G_T$ than G_S embeds in G_T as a combinatorial tree (disregarding the orders) and so $S \leq_{max} T$ by the second part of the previous proof. $\square$

Our aim now is to show that every analytic quasi-order is Borel bi-reducible to an embeddability relation on a Borel invariant class of ordered combinatorial trees. The key is to show that distinct normal trees S, T on $2 \times \omega$ give rise to non-isomorphic ordered combinatorial trees G_S, G_T and each G_S is rigid (i.e. has no nontrivial automorphism).

Lemma 29. *Let S, T be normal trees, and G_S and G_T the ordered combinatorial trees defined as in the previous proof. If $S \neq T$ then $G_S \ncong G_T$.*

Proof. Suppose i is an isomorphism between G_S and G_T. Since the orders $\leq_S$ and $\leq_T$ coincide on $\omega^{<\omega}$ we have that $i \restriction \omega^{<\omega}$ must be the identity. Suppose now $(u, s) \in S$: as in the earlier proof, the point $(u, s, 0^{2\theta(u)+2})$ must be sent to a point of the form $(u, i(s), 0^{2\theta(u)+2}) = (u, s, 0^{2\theta(u)+2})$, and the existence of such a point witnesses $(u, s) \in T$. Hence $S \subseteq T$. Exchanging the role of S and T and using i^{-1} instead of i one gets $T \subseteq S$, and therefore $S = T$. $\square$

Although the domain of each ordered combinatorial tree of the form G_T is formally different from ω, one can easily code such a structure into another structure $\hat{G}_T$ with domain ω. For simplicity we will identify the structures G_T and $\hat{G}_T$.

Let S_∞ be the Polish group of permutations on ω, $\mathcal{L} = \{P, Q\}$ the relational language with just two binary symbols and $j_\mathcal{L} \colon S_\infty \times \mathrm{Mod}_\mathcal{L} \to \mathrm{Mod}_\mathcal{L}$ the usual (continuous) action of S_∞ on $\mathrm{Mod}_\mathcal{L}$, the collection of all countable $\mathcal{L}$-structures, defined by sending $(p, \mathcal{A})$ to the isomorphic copy of $\mathcal{A}$ obtained by applying the permutation p to $\mathcal{A}$. For every normal tree S on $2 \times \omega$ and $p \in S_\infty$, put $G_{S,p} = j_\mathcal{L}(p, G_S)$, where G_S is the ordered combinatorial tree obtained from S as above (identified with a structure on ω).

Lemma 30. *For every distinct $p, q \in S_\infty$ and every normal tree S on $2 \times \omega$, we have $G_{S,p} \neq G_{S,q}$.*

Proof. Let $\leq_{S,p}$ and $\leq_{S,q}$ be the well-orders on $G_{S,p}$ and $G_{S,q}$, respectively. Let g be the $\leq_S$-minimal element of G_S such that $p(g) \neq q(g)$. We claim that $p(g) \leq_{S,p} q(g)$ but $p(g) \not\leq_{S,q} q(g)$ (this implies that the two structures $G_{S,p}$ and $G_{S,q}$ are different). Assume toward a contradiction that $q(g) <_{S,p} p(g)$ and therefore $p^{-1}(q(g)) <_S g$; then $q(p^{-1}(q(g))) <_{S,q} q(g)$. But as $p(p^{-1}(q(g))) = q(g)$, the previous inequality shows that $p(p^{-1}(q(g))) \neq q(p^{-1}(q(g)))$, contradicting the $\leq_S$-minimality of g. Therefore $p(g) \leq_{S,p} q(g)$.

Assume now towards a contradiction that $p(g) \leq_{S,q} q(g)$. Since $p(g) \neq q(g)$ (by hypothesis) we get $p(g) <_{S,q} q(g)$, which implies $q^{-1}(p(g)) <_S g$. Arguing as before (with p and q exchanged), we get a contradiction with the $\leq_S$-minimality of g. Therefore $p(g) \not\leq_{S,q} q(g)$, as required. □

Now we are ready to prove:

Theorem 31. *If R is an analytic quasi-order on 2^ω, then there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that R is Borel bi-reducible to embeddability on $Mod_\varphi = \{x \in Mod_\mathcal{L} \mid x \vDash \varphi\}$.*

Proof. For $x \in 2^\omega$ let S^x be defined as before, so that the map which sends x to S^x is Borel and injective. Let R' be the quasi-order on $X \times S_\infty$ defined by $(x, p) R' (y, q) \iff xRy$. It is clear that R and R' are Borel equivalent (as witnessed by the maps $x \mapsto (x, id)$ and $(x, p) \mapsto x$), hence it is enough to prove the theorem for R' (on the space $2^\omega \times S_\infty$). We will find a Borel and invariant subset Z of $\mathrm{Mod}_\mathcal{L}$ and an injective reduction of R' to the embeddability relation $\sqsubseteq$ (on $\mathrm{Mod}_\mathcal{L}$) with range Z, and then use the fact (due to Lopez-Escobar) that such a Z must coincide with Mod_φ for some $\mathcal{L}_{\omega_1\omega}$-sentence φ.

Using the same notation of the previous lemmas, consider the (Borel, indeed continuous) map f which sends (x,p) to $G_{S^x,p}$. First note that f reduces R' to the embedding relation $\sqsubseteq$, as

$$(x,p)R'(y,q) \iff xRy \iff S^x \leq_{max} S^y \iff G_{S^x} \sqsubseteq G_{S^y} \iff G_{S^x,p} \sqsubseteq G_{S^y,q}.$$

We now show that f is injective: Assume $(x,p) \neq (y,q)$. If $x \neq y$ then $S^x \neq S^y$, and therefore by Lemma 29 we get that $G_{S^x} \ncong G_{S^y}$, which in turn implies that $G_{S^x,p} \ncong G_{S^y,q}$ as well (so that, in particular, $G_{S^x,p}$ and $G_{S^y,q}$ must be different). If instead $x = y$ but $p \neq q$, then by Lemma 30 we get $G_{S^x,p} \neq G_{S^x,q} = G_{S^y,q}$ and hence we are done.

Since $X \times S_\infty$ is a Borel set and f is Borel and injective, we get that $f[X \times S_\infty] \subseteq \mathrm{Mod}_\mathcal{L}$ is a Borel set and that f^{-1} is Borel as well. But $f[X \times S_\infty]$ is clearly invariant under isomorphism, so $f[X \times S_\infty] = \mathrm{Mod}_\varphi$ for some $\mathcal{L}_{\omega_1\omega}$-sentence φ. Since f and f^{-1} witness the Borel bi-reducibility of R' and the embeddability relation on Mod_φ, this concludes the proof. □

Corollary 32. *Every analytic equivalence relation is Borel bi-reducible to a bi-embeddability relation on Mod(φ) for some sentence φ of $\mathcal{L}_{\omega_1\omega}$.*

13.-14.Vorlesungen

Topic 5: Computable Model Theory

We'll next discuss some recent work of Montalbán equating the Vaught Conjecture for infinitary sentences (i.e. sentences of $\mathcal{L}_{\omega_1\omega}$) with a statement in computable model theory, under a suitable set-theoretic hypothesis (PD, the axiom of Projective Determinacy).

Recall that the original Vaught Conjecture is the statement that if a first-order theory has uncountably many countable models (up to isomorphism) then it has continuum many. Here we replace "first-order theory" with "infinitary sentence" and take the Vaught Conjecture to be this stronger statement.

Notice that CH implies Vaught's Conjecture. But this misses the point, as in fact there is an absolute version of Vaught's Conjecture which is equivalent to Vaught's Conjecture when CH fails and which is not sensitive to CH. This is explained by the proof of the following theorem of Morley:

Theorem 33. *(Morley) If an infinitary sentence φ has more than $\aleph_1$ countable models then it has continuum many.*

Proof. For a countable ordinal α and countable structures $\mathcal{A}$, $\mathcal{B}$, write $\mathcal{A} \equiv_\alpha \mathcal{B}$ iff $\mathcal{A}$ and $\mathcal{B}$ satisfy the same infinitary sentences of rank less than α. (The rank of $\bigvee \Phi$ is the strict sup of the ranks of the $\varphi \in \Phi$, the ranks of $\forall x\varphi$, $\sim \varphi$ are the rank of φ $+1$, the rank of an atomic formula is 0.) The equivalence relations $\equiv_\alpha$ are Borel (on codes for countable structures). A Borel equivalence relation has either countably many or continuum many classes, so if φ has fewer than continuum many countable models, it follows that for each countable α, there are only countably many $\equiv_\alpha$ classes of countable models of φ. Now Scott's Theorem says that for any countable structure $\mathcal{A}$ the isomorphism type of $\mathcal{A}$ is determined by its $\equiv_\alpha$ class for α sufficiently large, approximately the "Scott rank" of $\mathcal{A}$. It follows that for any countable α there are at most countably many models of φ of "Scott rank" less than α and therefore there are at most $\aleph_1$ countable models of φ. □

A sentence φ is *scattered* if for each countable α, there are only countably many $\equiv_\alpha$ classes of countable models of φ. And φ is a *counterexample to Vaught's Conjecture* iff φ is scattered and has uncountably many countable models.

If φ is a counterexample to Vaught's Conjecture in this sense then by Lévy absoluteness, it is still a counterexample after we enlarge the universe of sets. In particular if we force to make CH false then we get a counterexample to Vaught's Conjecture in the original sense (i.e., a sentence with uncountably many but fewer than continuum many countable models). And conversely, any counterexample to Vaught's Conjecture in the original sense yields a counterexample in the above sense, by Morley's Theorem.

Now here is a statement of Montalbán's theorem, still using many notions that need to be explained.

Theorem 34. *(Montalbán) Assume PD and let φ be a sentence of $\mathcal{L}_{\omega_1\omega}$ with uncountably many countable models. Then the following are equivalent:*

(V1) φ is a counterexample to Vaught's conjecture (i.e. φ is scattered).
(V2) Relative to some oracle, φ satisfies Hyp is recursive.
(V3) Relative to some oracle,

$$\{Sp(\mathcal{A}) \mid \mathcal{A} \models \varphi\} = \{\{x \mid \omega_1^x \geq \alpha\} \mid \alpha < \omega_1\}.$$

A class of structures $\mathbb{K}$ satisfies *Hyp is recursive* iff every hyperarithmetic (i.e., Δ_1^1) structure in $\mathbb{K}$ has a computable copy. This is true for the class of wellorders and other classes (such as superatomic Boolean algebras) which are closely related to wellorders. None of these classes are axiomatisable by an infinitary sentence. $\mathbb{K}$ satisfies *Hyp is recursive relativ eto some oracle* iff for some real x and all $y \geq_T x$, every y-hyperarithmetic structure in $\mathbb{K}$ has a y-computable copy.

The *spectrum* $Sp(\mathcal{A})$ of a structure $\mathcal{A}$ is the set of all x such that $\mathcal{A}$ has an x-computable copy. For any x, ω_1^x is the least ordinal which is not x-computable, i.e., the least α such that $(\alpha, \in)$ has no x-computable copy. Using these definitions it is easy to check that (V3) implies (V2).

Note that for a wellorder $\mathcal{A}$ of ordertype α, $\mathrm{Sp}(\mathcal{A})$ is $\{x \mid \omega_1^x > \alpha\}$. So the class of wellorders does not yield the class of spectra exhibited in (V3). There is however a Σ_1^1 class of structures the spectra of whose elements are exactly those of the form $\{x \mid \omega_1^x \geq \alpha\}$, $\alpha < \omega_1$; an example is the class of linear orders of the form $\mathbb{Z}^\alpha \cdot \mathbb{Q}$, $\alpha < \omega_1$.

About Scott rank

Fix a countable structure $\mathcal{A}$. By induction on α define equivalence relations $\equiv_\alpha$ in finite tuples from A of the same length by:

$\vec{a} \equiv_0 \vec{b}$ iff $\vec{a}$, $\vec{b}$ satisfy the same atomic formulas in $\mathcal{A}$.
For limit λ, $\vec{a} \equiv_\lambda \vec{b}$ iff $\vec{a} \equiv_\alpha \vec{b}$ for all $\alpha < \lambda$.
$\vec{a} \equiv_{\alpha+1} \vec{b}$ iff for all a' there exists b' such that $\vec{a} * a' \equiv_\alpha \vec{b} * b'$ and conversely (for all b' there exists a' such that $\cdots$).

And for each tuple $\vec{a}$ define $\rho_{\mathcal{A}}(\vec{a})$ to be the least α such that $\vec{b} \equiv_\alpha \vec{a} \to \vec{b} \equiv_\beta \vec{a}$ for all β. The *Scott rank of $\mathcal{A}$, $SR(\mathcal{A})$* is the supremum of the $\rho_{\mathcal{A}}(\vec{a}) + 1$ for $\vec{a}$ a finite tuple from A. There is an infinitary sentence φ, called the *Scott sentence of* $\mathcal{A}$, such that φ has rank a bit more than $\mathrm{SR}(\mathcal{A})$ and the models of φ are exactly the isomorphic copies of $\mathcal{A}$.

Now suppose that $\mathcal{A}$ were a computable (or even Hyp) structure. Then for each $\vec{a}$ in A, $\rho_{\mathcal{A}}(\vec{a})$ is at most ω_1^{ck}, the least nonrecursive ordinal. Therefore the Scott rank of $\mathcal{A}$ is at most $\omega_1^{ck}+1$. There are computable structures of this highest possible Scott rank such as the Harrison order, which has ordertype $\omega_1^{ck}\cdot(1+\mathbb{Q})$. There are also examples of Scott rank ω_1^{ck}. We say that a computable structure $\mathcal{A}$ has *high Scott rank* iff its Scott rank is at least ω_1^{ck}.

For any countable structure $\mathcal{A}$ we can define $\omega_1^{\mathcal{A}}$ to be the min of the ω_1^x for x in the spectrum of $\mathcal{A}$. Then $\mathcal{A}$ has Scott rank at most $\omega_1^{\mathcal{A}}+1$ and $\mathcal{A}$ has *high Scott rank* iff its Scott rank is at least $\omega_1^{\mathcal{A}}$.

We now prepare to prove a special case of (V1) implies (V3) of the Theorem. We say that φ is a *minimal counterexample to Vaught's Conjecture* iff φ is a counterexample to Vaught's Conjecture and for any infinitary sentence ψ, either $\varphi\wedge\psi$ or $\varphi\wedge\sim\psi$ has countable many models.

If there is a counterexample to Vaught's Conjecture then there is a minimal one. The idea of the proof is to take a counterexample φ with no minimal strengthening and strengthen it to a perfect tree of counterexamples, yielding continuum many models of φ, contradicting the assumption that φ is scattered.

We will show that any minimal counterexample to Vaught's Conjecture satisfies (V3), using PD. First we discuss some consequences of PD due to Martin.

Martin's Lemmas

A *pointed tree* is a perfect subtree P of $2^{<\omega}$ all of whose (infinite) paths compute P. To each $x\in 2^\omega$ associate the path $P(x)$ through P obtained by following x at each split of P. When P is pointed, we have $x\oplus P\leq_T P(x)$ so $P(x)\equiv_T x\oplus P$. Thus the Turing degrees of the paths through P are exactly the Turing degrees of the reals which compute P. The set of such reals is called the *cone above* P.

For $A\subseteq 2^\omega$ the game $G(A)$ is played as follows: There are two players I, II. At even stages I plays a 0 or a 1 and at odd stages, II plays a 0 or a 1; the result is an element x of 2^ω. Then I wins this play of the game iff x

belongs to A. The *axiom of determinacy* asserts that for any A, one of the players has a winning strategy in this game (i.e., a rule for determining how to play next, given earlier plays). The *axiom of projective determinacy (PD)* is the same under the assumption that A is a subset of 2^ω which is *projective*, i.e., Σ^1_n for some n (equivalently, definable in second order arithmetic).

Lemma 35. *Assume PD and suppose that A is a projective set of reals which is Turing cofinal, i.e., for all x there is $y \in A$ with $x \leq_T y$. Then there is a pointed tree, all of whose paths belong to A.*

Proof. We prove it in the special case that A is *Turing invariant*, i.e., closed under $\equiv_T$.

Assume that I has a winning strategy in the game $G(A)$. A strategy for I can be though of as a function from finite strings of 0's and 1's of even length into 2; let $s \in 2^\omega$ be a real coding such a strategy. Now let P be the set of all initial segments of plays of the game $G(A)$ where I uses his strategy and II plays $s(n)$ at stage $4n+1$ (and II plays anything he wants at stages of the form $4n+3$). Then P is a perfect tree (with splitting nodes at lengths $4n+3$ for some n) and P is pointed because if x is a path through P then $s(n) = x(4n+1)$ and hence I's strategy as well as the tree P is computable from x.

Finally we show that II cannot have a winning strategy and therefore by the assumption of PD we are done. Otherwise, using the above argument, we get a pointed tree P all of whose paths are not in A; but then each Turing degree in the cone above P contains a real not in A and since A is closed under $\equiv_T$, it follows that A contains no real Turing above P, contrary to the hypothesis that A is Turing cofinal. $\square$

15.Vorlesung

Last time we proved Lemma 35 in the special case where A is *Turing invariant*, i.e., closed under $\equiv_T$. Now we give the proof in the general case.

Recall that the game $G(A)$ is played as follows: There are two players, I and II. At even stages I plays a 0 or a 1 and at odd stages, II plays a 0 or a 1; the result is an element x of 2^ω. Then I wins this play of the game iff x belongs to A.

We generalise the game $G(A)$ as follows: Suppose that T is a perfect subtree of $2^{<\omega}$ and define a bijection π_T from the full tree $2^{<\omega}$ onto the splitting nodes of T by:

$\pi_T(\emptyset) =$ the least splitting node of T
$\pi_T(s * i) =$ the least splitting node of T extending $\pi_T(s) * i$.

Then the game $G(A, T)$ is played just like the game $G(A)$ but the winning condition is: If $x \in 2^\omega$ is the result of the play then I wins iff the union of the $\pi_T(x \restriction n)$'s belongs to A.

Now proceed as follows. Let A^* denote the closure of A under $\equiv_T$, i.e. $\{x \mid x \equiv_T y$ for some $y \in A\}$. Then II cannot have a winning strategy in $G(A^*)$ because otherwise there is a pointed tree, all of whose branches belong to the complement of A^* and therefore all sufficiently large Turing degrees contain an element of the complement of A^*; but his implies that A is not Turing cofinal. So by PD, I has a winning strategy in $G(A^*)$. It follows that there is a pointed tree, all of whose branches belong to A^*.

For each $n \in \omega$, view n as a pair (n_0, n_1) and define:

$A_n = \{x \mid \{n_0\}^x$ is total, $\{n_0\}^x$ belongs to A and $\{n_1\}^{\{n_0\}^x} = x\}$.

Note that A^* is the union of the A_n's.

If I has a winning strategy in one of the games $G(A_n)$ then A contains the branches of a pointed tree: If σ is a winning strategy for I then as before consider the tree T_n of all plays of the game where I uses the strategy σ and at stage $4n + 1$, II plays $c(n)$, where $c \in 2^\omega$ is a code for σ; then T_n is a pointed tree all of whose branches belong to A_n. But then $\{\{n_0\}^x \mid x$ is a branch through $T_n\}$ consists of the branches through another pointed tree T, all of whose branches belong to A.

More generally, if T is a pointed tree and I has a winning strategy in one of the games $G(A_n, T)$ then A contains the branches of a pointed tree and we are done.

Now inductively define pointed trees T_n as follows: T_0 is a pointed tree, all of whose branches belong to A^*. Suppose that T_n is defined. Then consider the game $G(A_n, T_n)$. As said above, if I has a winning strategy in this game then A contains the branches of a pointed tree and we are done. Let σ_n be a winning strategy for II in this game and let $c \in 2^\omega$ be a code for σ_n. Now take T'_{n+1} to consist of all finite plays of the game $G(A_n, T_n)$ in which II uses the strategy σ_n and I plays $c(k)$ at stage $4k+1$. Define T_{n+1} to be the subtree of T_n whose splitting nodes are the nodes $\pi_{T_n}(s)$ for $s \in T'_{n+1}$. Then T_{n+1} is a pointed tree, all of whose branches belong to the complement of A_n.

The intersection of the T_n's contains an infinite branch x. But then x belongs to A^* but not to any A_n, contradiction! $\square$

16.-17.Vorlesungen

Lemma 36. *Assume PD and suppose $f : 2^\omega \to \omega_1$ is Turing invariant (i.e., $x \equiv_T y \to f(x) = f(y)$) and has a projective presentation, i.e., for some projective $g : 2^\omega \to 2^\omega$, $g(x)$ is a wellorder of length $f(x)$ for each x. If for each x, $f(x) < \omega_1^x$ then f is constant on a cone (i.e., constant on all reals Turing above some fixed real).*

Proof. For each x there is an $e \in \omega$ such that $\{e\}^x$ is a wellorder of length $f(x)$. Therefore there is some fixed e such that on a Turing cofinal set A of x's, $\{e\}^x$ has length $f(x)$. By the previous lemma there is a pointed tree P such that all paths x through P satisfy that $g(x)$ is isomorphic to $\{e\}^x$. So $x \mapsto \{e\}^x$ restricted to the paths through P is a continuous map from a closed set to the set of wellorders and therefore is bounded below ω_1. It follows that f is bounded below some $\alpha < \omega_1$ on the cone above P. Then f is constant on a Turing cofinal set of reals so again by the previous lemma, f is constant on a cone. $\square$

Now we apply Martin's lemma to prove a result about ranked equivalence relations. These consist of an equivalence relation $\equiv$ on a subset R of 2^ω together with an $\equiv$-invariant function $r : R \to \omega_1$. We say that $(R, \equiv, r)$ is *projective* if R and $\equiv$ are projective and r has a projective represenation, i.e. there is a projective function $r^* : 2^\omega \to 2^\omega$ such that $r^*(x)$ is a real coding the ordinal $r(x)$ for each x.

$(R, \equiv, r)$ is *scattered* if for each $\alpha < \omega_1$ there are only countable many $\equiv$ classes of reals x such that $r(x) = \alpha$. A key example is where r assigns Scott rank to the models of a scattered theory and $\equiv$ is the equivalence relation of isomorphism on those models.

Lemma 37. *Assume PD and suppose that $(R, \equiv, r)$ is a scattered projective ranked equivalence relation. Then for a cone of z's, we have that if $r(x) < \omega_1^z$ then z computes a member of $[x]_\equiv$, the $\equiv$ equivalence class of x.*

Proof. Otherwise on every cone there is a z for which there is an x with $r(x) < \omega_1^z$ and the $[x]_\equiv$ has no z-computable member. There is a cone of such z's. For each z in this cone let α_z be the least $\alpha < \omega_1^z$ for which there is an x such that $r(x) = \alpha$ and z does not compute any member of $[x]_\equiv$. By the previous lemma, the function $z \mapsto \alpha_z$ is constant on a cone. But this is not possible because there are only countably many $\equiv$ classes to which r assigns a rank less than α and so any z of sufficiently large Turing degree can compute members of all of them. □

The previous result has some nice applications.

Example 1. Consider isomorphism of countable wellorders where the rank function is the ordertype. It follows from the above that for a cone of z's, any wellorder which is Hyp in z has a z-computable copy. Of course Spector proved that this is true for all z.

Example 2. Consider bi-embeddability of linear orders. A linear order is *scattered* if it does not embed $\mathbb{Q}$; the non-scattered linear orders form a single equivalence class under bi-embeddability. To each scattered linear order is associated its *Hausdorff rank*, the least α such that $\mathcal{L}$ embeds into $\mathbb{Z}^\alpha$ (finite-supported functions from α into $\mathbb{Z}$, ordered by comparing the largest place of difference). For each scattered $\mathcal{L}$, the Hausdorff rank of $\mathcal{L}$ is less than $\omega_1^{\mathcal{L}}$. Laver showed that there are only countably many bi-embeddability classes of each Hausdorff rank. So by the above result, for cone of z's, any scattered linear order which is Hyp in z is bi-embeddable with a z-computable such order. Montalbán proved that this is true for all z.

Example 3. Consider bi-embeddability on countable p-groups. The associated ranking function is called *Ulm rank* and Barwise-Eklof showed that there are only countably many equivalence classes of each Ulm rank. Except for one class, represented by the group $\mathbb{Z}(p^\infty)^\omega$, the Ulm rank of a group G is less than ω_1^G. It follows then from the above result that for a cone of z's, every p-group which is Hyp in z is bi-embeddable with a z-computable such group. Greenberg and Montalbán showed that this is true for all z.

The family of spectra of minimal counterexamples

Recall that φ is a *minimal counterexample* to Vaught's conjecture if φ is scattered (i.e. has only countably many models of any given Scott rank), has uncountably many models but for each ψ, either $\varphi \wedge \psi$ or $\varphi \wedge \sim \psi$ has countably many models. If there is a counterexample to Vaught's conjecture then there is a minimal one.

Also recall that the *spectrum* of a model $\mathcal{A}$, $\mathrm{Sp}(\mathcal{A})$, is the set of z for which $\mathcal{A}$ has a z-computable copy. And $\omega_1^{\mathcal{A}}$ denotes the least ω_1^z for $z \in \mathrm{Sp}(\mathcal{A})$.

More generally, for any real z_0 define the z_0-*spectrum* of $\mathcal{A}$, $\mathrm{Sp}^{z_0}(\mathcal{A})$, to be the set of $z \geq_T z_0$ such that $\mathcal{A}$ has a z-computable copy. And $\omega_1^{\mathcal{A},z_0}$ denotes the least ω_1^z for $z \in \mathrm{Sp}^{z_0}(\mathcal{A})$.

Theorem 38. *(Montalbán) Assume PD and suppose that φ is a minimal counterexample to Vaught's conjecture. Then for some z_0 and every model $\mathcal{A}$ of φ:*

$$Sp^{z_0}(\mathcal{A}) = \{z \geq_T z_0 \mid \omega_1^z \geq \omega_1^{\mathcal{A},z_0}\}.$$

Corollary 39. *Under the hypotheses above, there is a z_0 such that for every model $\mathcal{A}$ of φ, if $z \geq_T z_0$ and there is a copy of $\mathcal{A}$ which is Hyp in z then there is also a copy of $\mathcal{A}$ which is z-computable.*

Proof of Corollary. If $\mathcal{A}$ has a copy which is Hyp in z then $\mathcal{A}$ has a copy which is computable in z^* where $z^* \geq_T z$ is Hyp in z; but as $\omega_1^{z^*} = \omega_1^z$ it follows from the Theorem that $\mathcal{A}$ also has a copy which is computable in z. $\square$

Proof of Theorem. Note that the direction $\subseteq$ of the Theorem is immediate from the definitions. Thus we want to show that relative to some real, if $\mathcal{A}$ is a model of φ and ω_1^z is at least $\omega_1^{\mathcal{A}}$, then z computes a copy of $\mathcal{A}$. There will be two cases: $\mathcal{A}$ has Scott rank $< \omega_1^z$ and $\mathcal{A}$ has Scott rank $\geq \omega_1^z$; note that in the latter case, as the Scott rank of $\mathcal{A}$ is at most $\omega_1^{\mathcal{A}} + 1$, we have that $\omega_1^z = \omega_1^{\mathcal{A}}$.

For the models $\mathcal{A}$ of Scott rank $< \omega_1^z$, we use our earlier work on scattered ranked equivalence relations: Consider $(\mathrm{Mod}(\varphi), \simeq, \mathrm{SR})$ where $\mathrm{Mod}(\varphi)$ denotes the countable models of φ, and SR denotes Scott rank. As φ is a counterexample to Vaught's conjecture, this is a (projective) scattered equivalence relation. So by an earlier result, relative to some real, if $\mathcal{A}$ has Scott rank $< \omega_1^z$ then z can compute a copy of $\mathcal{A}$.

Now to show that relative to some real, if $\mathcal{A}$ is a model of φ with $\omega_1^z = \omega_1^{\mathcal{A}} \leq \mathrm{SR}(\mathcal{A})$ then z can compute a copy of $\mathcal{A}$ it suffices to show that relative to some real:

$$X = \{z \mid z \text{ computes a copy of all } \mathcal{A} \in \mathrm{Mod}(\varphi) \text{ with } \omega_1^z = \omega_1^{\mathcal{A}} \leq \mathrm{SR}(\mathcal{A})\}$$

is Turing cofinal. For given this, we can apply PD to conclude that relative to some real, z computes a copy of all $\mathcal{A}$ in $\mathrm{Mod}(\varphi)$ with $\omega_1^z = \omega_1^{\mathcal{A}} \leq \mathrm{SR}(\mathcal{A})$, so we are done.

So we show that relative to some real, X is Turing cofinal.

First we show that for any z there is $z^* \geq_T z$ such that z^* computes some $\mathcal{A} \in \mathrm{Mod}(\varphi)$ with $\mathrm{SR}(\mathcal{A}) \geq \omega_1^{\mathcal{A}} = \omega_1^{z^*}$: This follows from Gandy's Basis Theorem as follows. We may assume that φ is computable relative to z. Let α be ω_1^z. As φ has uncountably many models and is scattered, there are models of φ of arbitrarily high Scott rank and therefore there is one of Scott rank at least α. The set of codes for models of Scott rank at least α is a Σ_1^1 set with parameter z and by Gandy's Basis Theorem there is therefore a code $z^* \geq_T z$ for such a model satisfying $\omega_1^{z^*} = \omega_1^z$; then the model $\mathcal{A}$ coded by z^* satisfies $\mathrm{SR}(\mathcal{A}) \geq \omega_1^{\mathcal{A}} = \alpha$.

Apply PD to conclude that relative to some z_0, every real z computes at least one $\mathcal{A} \in \mathrm{Mod}(\varphi)$ of high Scott rank such that $\omega_1^{\mathcal{A}} = \omega_1^z$.

Now we show that X is Turing cofinal relative to some real. We first need a lemma.

Lemma 40. *There is a club C in ω_1 such that for each $\alpha \in C$, if two models of φ have Scott rank at least α then they are elementarily equivalent for sentences of rank less than α.*

Proof. For a countable ordinal α let $\equiv_\alpha$ denote the equivalence relation on $\mathrm{Mod}(\varphi)$ of satisfying the same sentences of rank less than α. Each $\equiv_\alpha$ equivalence class forms the models of a sentence of rank at most $\alpha + 1$. And as φ is a minimal counterexample to Vaught's Conjecture, at most one $\equiv_\alpha$ class is uncountable (up to isomorphism). Let $f(\alpha)$ bound the Scott ranks of the models in the $\equiv_\alpha$ classes with only countably many models and suppose that β is closed under the function f. Then if two models of φ have Scott rank at least β then they belong to the same $\equiv_\alpha$ class for each $\alpha < \beta$ and therefore satisfy the same sentences of rank less than β. Let C be the club of such β's. $\square$

18.-19.Vorlesungen

Now let $z \geq_T z_0$ be any real and choose an ordinal α in the club C of the Lemma of the form $\omega_1^{z^*}$ for some $z^* \geq_T z$. (This is possible as C contains z-admissible ordinals and Sacks showed that any z-admissible ordinal is of this form.) We claim that z^* belongs to X (relativised to z_0). Indeed, by choice of z_0, z^* computes at least one $\mathcal{A}$ in $\mathrm{Mod}(\varphi)$ of high Scott rank $\geq \omega_1^{z^*}$. Let $\mathcal{B}$ be another model in $\mathrm{Mod}(\varphi)$ of high Scott rank $\geq \omega_1^{z^*} = \omega_1^{\mathcal{B}}$; we claim that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic: Choose $z^{**} \geq_T z_0$ which computes $\mathcal{B}$ with $\omega_1^{z^{**}} = \omega_1^{z^*}$. Then there is an interpolating z^{***} such that

$$\alpha = \omega_1^{z^*} = \omega_1^{z^*, z^{***}} = \omega_1^{z^{***}} = \omega_1^{z^{***}, z^{**}}.$$

But z^{***} also computes a model $\mathcal{C}$ of φ of high Scott rank with $\omega_1^{\mathcal{C}} = \alpha$. By choice of α, the three models $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are equivalent for sentences of rank less than α. But as $\omega_1^{z^*, z^{***}} = \alpha$ it follows that $\mathcal{A}$ and $\mathcal{C}$ are isomorphic; similarly $\mathcal{B}$ and $\mathcal{C}$ are isomorphic, so $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Therefore z^* belongs to X (relativised to z_0), as desired. $\square$

Topic 6: The effective theory of Borel equivalence relations

If E and F are Borel equivalence relations then E is *Borel reducible* to F if and only if there is a Borel function $f : X \to Y$ such that xEy if and only if $f(x)Ff(y)$. The study of Borel equivalence relations under Borel reducibility has developed into a rich area of descriptive set theory. In this non-effective setting, Borel equivalence relations with countably many equivalence classes are equivalent (i.e. bi-reducible) exactly if they have the same number of equivalence classes. For Borel equivalence relations with uncountably many equivalence classes there are two fundamental dichotomies:

The Silver Dichotomy. If E is a Borel equivalence relation with uncountably many equivalence classes then equality on $\mathcal{P}(\omega)$, the power set of ω, is Borel reducible to E.

The Harrington-Kechris-Louveau Dichotomy. If E is a Borel equivalence relation not Borel reducible to equality on $\mathcal{P}(\omega)$ then E_0 is Borel reducible to E, where E_0 is equality modulo finite on $\mathcal{P}(\omega)$.

Now we introduce the effective version of this theory. If E and F are effectively Borel (i.e., Δ^1_1) equivalence relations then we say that E is *effectively Borel reducible* to F if there is an effectively Borel function $f : X \to Y$ such that xEy if and only if $f(x)Ff(y)$. The resulting effective theory reveals an unexpectedly rich new structure, even for equivalence relations with finitely many classes. For $n \leq \omega$, let $=_n$ denote equality on n, let $=_{\mathcal{P}(\omega)}$ denote equality on the power set of ω and let E_0 denote equality modulo finite on $\mathcal{P}(\omega)$. The notion of effectively Borel reducibility on effectively Borel equivalence relations naturally gives rise to a degree structure, which we denote by $\mathcal{H}$.

We'll show the following:

Theorem 41. *(Katia-Asger-Sy) For any finite n, the partial order of Δ^1_1 subsets of ω under inclusion can be order-preservingly embedded into $\mathcal{H}$ between the degrees of $=_n$ and $=_{n+1}$. The same holds between the degrees of $=_\omega$ and $=_{\mathcal{P}(\omega)}$, and between $=_{\mathcal{P}(\omega)}$ and E_0.*

A basic tool in the proof of this theorem is the following result:

$(*)$ *There are effectively Borel sets A and B such that for no effectively Borel function f does one have $f[A] \subseteq B$ or $f[B] \subseteq A$.*

$(*)$ is proved via a Barwise compactness argument applied to a deep result of Harrington establishing for any recursive ordinal α the existence of Π^0_1 singletons whose α-jumps are Turing incomparable.

Harrington's proof of the Silver dichotomy and the original proof of the Harrington-Kechris-Louveau dichotomy respectively show that if an effectively Borel equivalence relation has countably many equivalence classes then it is effectively Borel reducible to $=_\omega$ and if it is Borel reducible to $=_{\mathcal{P}(\omega)}$ then it is in fact effectively Borel reducible to $=_{\mathcal{P}(\omega)}$.

Theorem 42. *Let O denote Kleene's O. If an effectively Borel equivalence relation E has uncountably many equivalence classes then there is a $\Delta^1_1(O)$ function reducing $=_{\mathcal{P}(\omega)}$ to E, and this parameter is best possible. If an effectively Borel equivalence relation E is not Borel reducible to $=_{\mathcal{P}(\omega)}$ then there is a $\Delta^1_1(O)$ function reducing E_0 to E, and this parameter is best possible.*

In other words, while the first theorem rules out that the dichotomy Theorems of Silver and Harrington-Kechris-Louveau are effective, the second theorem shows that the Borel reductions obtained in the dichotomy Theorems can in fact be witnessed by $\Delta^1_1(O)$ functions, and that Kleene's O is the best possible parameter we can hope for in general. The proof of the second theorem is based on a detailed analysis of the effectiveness of category notions in the Gandy-Harrington topology, due to Asger Törnquist.

Harrington's Theorem and $(*)$

We will use the following deep result of Harrington without proof.

Theorem 43. *For any recursive ordinal* α *there is a sequence of reals* $\langle a_n | n < \omega\rangle$ *such that for some recursive sequence* $\langle \varphi_n | n < \omega\rangle$ *of* Π^0_1 *formulas,* a_n *is the unique solution to* φ_n *for each* n *and no* a_n *is recursive in the* α*-jump of* $\langle a_m \mid m \neq n\rangle$*.*

Using this we show:

Theorem 44. *There exist two nonempty* Π^0_1 *sets* A, B*, such that for no Hyp function* F *do we have* $F[A] \subseteq B$ *or* $F[B] \subseteq A$*.*

Proof. Let $\mathcal{A} = L_{\omega_1^{ck}}$, $\mathcal{L} \supseteq \{\in, <, \underline{x_0}, \underline{x_1}\} \cup \{\underline{\alpha} : \alpha \in \mathcal{A}\}$, where $\underline{x_0}, \underline{x_1}$ and $\underline{\alpha}$ are constant symbols. Consider the set of sentences Φ consisting of:

1. ZF^-, where ZF^- is ZF without Power Set,
2. $(\forall x)(x \in \underline{\omega} \leftrightarrow \bigvee_n x = \underline{n})$
3. $< = \in \restriction$ Ordinals
4. $\underline{x_0}, \underline{x_1} \subseteq \underline{\omega}$
5. $\bigvee_{\varphi \in \Pi^0_1} \left[(\exists! v)\varphi(v) \wedge \varphi(\underline{x_i})\right]$ ($i = 0, 1$, φ ranges over all Π^0_1 formulas.)
6. $\underline{x_0} \not\leq_T \underline{x_1}^{\underline{\alpha}}, \underline{x_1} \not\leq_T \underline{x_0}^{\underline{\alpha}}$, for all $\alpha < \omega_1^{ck}$.

The set Φ is a Σ_1 set of sentences. By Harrington's Theorem, for every recursive ordinal α there exist Π^0_1 singletons a_α, b_α, such that a_α is not recursive in the α-th Turing jump of b_α and b_α is not recursive in the α-th Turing jump of a_α. We apply Barwise Compactness to get a model $\langle M, E, <, x_0, x_1\rangle \models \Phi$ such that $L_{\omega_1^{ck}} \subseteq M$, M has nonstandard ordinals and every standard ordinal of M is recursive, i.e., the standard part of $<^M$ is ω_1^{ck}. Then in M there must be Π^0_1 singletons a and b such that $a \not\leq_T b^\alpha, b \not\leq_T a^\alpha$ for $\alpha < \omega_1^{ck}$ and since $\omega_1^a = \omega_1^b = \omega_1^{ck}$, a and b are Hyp-incomparable.

Choose Π^0_1 formulas φ_a and φ_b, such that in M, $\varphi_a(x) \leftrightarrow x = a$ and $\varphi_b(x) \leftrightarrow x = b$. Then the formulas φ_a and φ_b define Π^0_1 sets (not singletons) in V. Let $A = \{x : \varphi_a(x)\}$ and $B = \{x : \varphi_b(x)\}$. To finish the proof we show:

Claim. There is no Hyp function F such that $F[A] \subseteq B$ or $F[B] \subseteq A$.

Proof of Claim. By symmetry it suffices to prove that there is no Hyp function F such that $F[A] \subseteq B$. Suppose F were such a function. Consider $F(a)$; it is Hyp in a and therefore belongs to M. But by assumption $F(a)$ belongs to the Π^0_1 set B and therefore by definition of B, $\varphi_b(F(a))$ is true. But φ_b is a Π^0_1 formula and therefore $\varphi_b(F(a))$ also holds in M. It follows that $F(a) = b$. But then b is Hyp in a, implying that it is recursive in a^α for some $\alpha < \omega_1^{ck}$, contradicting the properties of a and b. □

Using the more general form of Harrington's theorem We also have:

Theorem 45. *There exists a uniform sequence $A_0, A_1, \ldots$ of nonempty Π^0_1 sets such that for each n there is no Hyp function F such that $F[A_n] \subseteq \bigcup_{m \neq n} A_m$.*

Hyp Equivalence Relations under Hyp Reducibility

Let E and F be equivalence relations on reals. We say that E is *Hyp-reducible* to F if there exists a Hyp function f such that xEy iff $f(x)Ff(y)$, in which case we will write $E \leq_H F$.

This induces a natural notion of *Hyp-equivalence* (or *Hyp bi-reducibility*) and *Hyp-degrees*: we let $E \equiv_H F$ if and only if $E \leq_H F$ and $F \leq_H E$.

For $0 < n < \omega$, let $=_n$ be the Hyp-degree of the equivalence relation: $x \equiv y \iff x(0) = y(0)$ or both $x(0), y(0) \geq n-1$. The Hyp-degree $=_\omega$ is the Hyp-degree of the equivalence relation $x \equiv y \iff x(0) = y(0)$.

Hyp Equivalence Relations with countably many classes

Proposition 46. *Let $1 \leq n \leq \omega$ and let E be a Hyp equivalence relation. Then $=_n \leq_H E$ iff E has at least n classes containing Hyp reals.*

Proof. ($\Rightarrow$): For every $1 \leq n \leq \omega$, the equivalence relation $=_n$ has exactly n equivalence classes and each of them contains a Hyp real. Under Hyp-reducibility Hyp reals are sent to Hyp reals and inequivalent reals are sent to inequivalent reals.
($\Leftarrow$): If n is finite, pick n Hyp reals $x_0, \ldots, x_{n-1}$ that lie in different equivalence classes of E. The function F that sends the i-th equivalence class of $=_n$ to x_i witnesses the reduction. Now suppose $n = \omega$. Suppose E is an equivalence relations with infinitely many classes containing Hyp reals. We want to prove that $=_\omega$ Hyp-reduces to E. We will find a Hyp sequence of equivalence classes of E with Hyp reals in them. Consider the following relation $P(X, Y)$ on $\omega \times (\omega^\omega)^{<\omega}$:

$$P(X,Y) \iff [X = (n, X_0, \ldots, X_n) \wedge \bigwedge_{i \neq j} \neg X_i E X_j] \longrightarrow$$
$$[Y = (n+1, Y_1, \ldots, Y_n, Y_{n+1}) \wedge \bigwedge_i X_i = Y_i \wedge \bigwedge_{i \neq j} \neg Y_i E Y_j]$$

Then P is Hyp. Moreover, as E has infinitely many Hyp classes, for every Hyp X there exists a Hyp Y such that $P(X, Y)$. It follows from Hyp Dependent Choice that there exists a uniform sequence of Hyp sets $X_0, X_1, \ldots$ such that

$$\forall i, j (i \neq j \to \neg X_i E X_j).$$

Then the function that sends the equivalence class $\{x : x(0) = n\}$ of $=_\omega$ to X_n is Hyp and witnesses the reduction. □

Corollary 47. *If $=_n \leq_H E$, for all $1 \leq n < \omega$, then $=_\omega \leq_H E$.*

Proposition 48. *Let $1 \leq n \leq \omega$ and let E be a Hyp equivalence relation. Then $E \leq_H =_n$ iff E has at most n classes.*

Proof sketch. The direction ($\Rightarrow$) is obvious since non-equivalent reals are sent to non-equivalent reals under Hyp-reducibility. To prove ($\Leftarrow$) we need to show that the equivalence classes of a Hyp equivalence relation with at most countably many equivalence classes are uniformly Hyp.

By Harrington's proof of the Silver Dichotomy, if E has only countably many classes then every real belongs to a Hyp subset of some equivalence class. Let C be the set of codes for Hyp subsets of an equivalence class; then C is Π^1_1. Consider the relation

$$R = \{(x, c) \mid c \in C \text{ and } x \in H(c), \text{ the Hyp set coded by } c\}.$$

Then R is Π^1_1 and can be uniformised by a Π^1_1 function F. As the values of F are numbers, F is Hyp and by separation we can choose a Hyp $D \subseteq C$, $D \supseteq \rangle(F)$. Now define an equivalence relation E^* on D by:

$$\begin{aligned} d_0 E^* d_1 &\iff (\forall x_0, x_1)(x_0 \in H(d_0) \wedge x_1 \in H(d_1)) \rightarrow x_0 E x_1 \\ &\iff (\exists x_0, x_1)(x_0 \in H(d_0) \wedge x_1 \in H(d_1) \wedge x_0 E x_1). \end{aligned}$$

i.e. $d_0 E^* d_1$ if and only if $H(d_0)$ and $H(d_1)$ are subsets of the same E-equivalence class. Note that E^* is Hyp. The relation E Hyp-reduces to E^* via $x \mapsto F(x)$. But E^* is just a Hyp relation on a Hyp set of numbers, so E^* is Hyp-reducible to $=_\omega$ (to see this, send c to the least number c^*, cE^*c^*).

Thus if E is a Hyp equivalence relation with at most countably many classes then E is Hyp-reducible to $=_\omega$. (In particular, all equivalence classes of E are Hyp.) One can similarly see that if E has at most n classes then E is Hyp-reducible to $=_n$. □

Obviously, the degree $=_1$ is Hyp-reducible to any equivalence relation. But $=_2$, the equivalence relation with the two classes $\{x : x(0) = 0\}$ and $\{x : x(0) \geq 1\}$ is *not* the successor to $=_1$. This is the content of the next theorem.

Theorem 49. 1. *There is a Hyp equivalence relation strictly between $=_1$ and $=_2$.*

2. *For every finite n, there is a Hyp equivalence relation strictly between $=_n$ and $=_{n+1}$.*

3. *For every $n_0 < n_1 \leq \omega$, there is a Hyp equivalence relation above $=_{n_0}$, below $=_{n_1}$ and incomparable with $=_n$, for all $n_0 < n < n_1$.*

Proof. There is a nonempty Hyp set X which contains no Hyp reals. Take a Hyp equivalence relation E with the two equivalence classes X and $\sim X$. By Proposition 63, E Hyp-reduces to $=_2$. By Proposition 61, $=_2$ does not Hyp-reduce to E.

To prove the second statement, we let E consist of exactly $n+1$ equivalence classes, such that only n of them contain Hyp reals: For each $i < n-1$, we define the i-th equivalence class by taking all $x \in \sim X$, such that $x(0) = i$. We take the n-th class to contain all $x \in \sim X$ with $x(0) \geq n-1$. And the $(n+1)$-st class is X.

For the proof of the third statement, consider an equivalence relation with n_1 classes such that only n_0 of them contain Hyp reals. □

Theorem 50. *There are incomparable Hyp equivalence relations between $=_1$ and $=_2$.*

Proof. We consider the following equivalence relations: Let A and B be Π^0_1 sets such that for no Hyp F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$. We take the equivalence relation E_A with two equivalence classes $A, \sim A$ and E_B with two equivalence classes $B, \sim B$. Then E_A and E_B are Hyp-reducible to $=_2$. By the properties of A and B, the relations E_A and E_B are Hyp-incomparable, as otherwise (since neither A nor B contain Hyp reals) we would have a Hyp function which maps A to B or vice versa. $\square$

Theorem 51. *The partial order of Hyp subsets of ω under inclusion can be order-preservingly embedded into the structure of degrees of Hyp equivalence relations between $=_1$ and $=_2$.*

Proof. Let X be a Hyp subset of ω. Define the corresponding equivalence relation E_X in the following way. We let xE_Xy iff both $x, y \in \bigcup_{i\in X} A_i$ or both $x, y \in \sim \bigcup_{i\in X} A_i$, where $A_0, A_1, \ldots$ are the sets constructed earlier using Harrington's theorem, i.e., they are uniformly Π^0_1 and no A_i can be mapped by a Hyp function into the union of the A_j's, j different from i. We check that $X \subseteq Y \iff E_X \leq_H E_Y$.

Suppose $X \subseteq Y$. For every $i \in X$ we send A_i into itself. We send $\sim \bigcup_{i\in X} A_i$ into a single Hyp real chosen in $\sim \bigcup_{i\in Y} A_i$. Therefore $E_X \leq_H E_Y$.

Now suppose $X \nsubseteq Y$ but $E_X \leq_H E_Y$ via a Hyp function F. Note that neither $\bigcup_{i\in X} A_i$ nor $\bigcup_{i\in Y} A_i$ contain Hyp reals. Thus F sends $\sim \bigcup_{i\in X} A_i$ to $\sim \bigcup_{i\in Y} A_i$ and $\bigcup_{i\in X} A_i$ to $\bigcup_{i\in Y} A_i$. Choose an $i_0 \in X \setminus Y$. Then $F[A_{i_0}] \subseteq \bigcup_{i\in Y} A_i \subseteq \bigcup_{i\neq i_0} A_i$, contradicting the properties of the sequence $A_0, A_1 \ldots$ $\square$

Corollary 52. *1. There are infinite antichains between $=_1$ and $=_2$.*

2. There are infinite descending chains between $=_1$ and $=_2$.

3. There are infinite ascending chains between $=_1$ and $=_2$.

The same proof shows:

Corollary 53. *For any $1 \leq n_0 < n_1 \leq \omega$ there is an embedding of the partial order $(\mathcal{P}(\omega) \cap Hyp, \subseteq)$ into the structure of degrees of Hyp equivalence relations that are above $=_{n_0}$, below $=_{n_1}$ and incomparable with each $=_n$ for $n_0 < n < n_1$.*

Hyp Equivalence Relations between $=_\omega$ and $=_{\mathcal{P}(\omega)}$

Let $=_{\mathcal{P}(\omega)}$ denote the Hyp-degree of the equivalence relation of $=$ on $\mathcal{P}(\omega)$. By an earlier Proposition and Silver's dichotomy, every Hyp equivalence relation E is either Hyp reducible to $=_\omega$, or $=_{\mathcal{P}(\omega)}$ is Borel reducible to E. We show that the latter option is not effective:

Theorem 54. *There exist Hyp-incomparable Hyp equivalence relations between $=_\omega$ and $=_{\mathcal{P}(\omega)}$.*

Proof. Suppose that A and B are Π^0_1 sets such that there is no Hyp function F such that $F[A] \subseteq B$ or $F[B] \subseteq A$.

Now consider the equivalence relations E_A and E_B:

$$xE_Ay \iff [(x \in A \wedge x = y) \vee (x, y \notin A \wedge x(0) = y(0))]$$

and similarly for E_B with B replacing A.

By sending n to the real $(n, 0, 0, \ldots)$ we get a Hyp reduction $=_\omega$ to E_A and E_B. Also E_A Hyp-reduces to $=_{\mathcal{P}(\omega)}$ via the map $G(x) = x$ if x belongs to A, $G(x) = (x(0), 0, 0, \ldots)$ for $x \notin A$. Similarly for B.

There is no Hyp reduction of E_A to E_B. Indeed, suppose that F were such a reduction and let C be the preimage under F of $\sim B$. As $\sim B$ is Σ^0_1, C is Hyp and therefore $A \cap C$ is also Hyp. But $A \cap C$ must be countable as F is a reduction. So if $A \cap C$ were nonempty it would have a Hyp element, contradicting the fact that A has no Hyp elements. Therefore F maps A into B, which is impossible by the choice of A and B. □

Theorem 55. *The partial order of Hyp subsets of ω under inclusion can be embedded into the structure of degrees of Hyp equivalence relations between $=_\omega$ and $=_{\mathcal{P}(\omega)}$.*

Proof. Let $A_0, A_1, \ldots$ be the uniformly Π^0_1 sets used earlier. For every Hyp set $X \subseteq \omega$ consider the equivalence relation

$$xE_Xy \iff [(x \in \bigcup_{i\in X} A_i \text{ and } x = y) \text{ or } (x, y \notin \bigcup_{i\in X} A_i \text{ and } x(0) = y(0))].$$

Then $=_\omega \leq_H E_X \leq_H =_{\mathcal{P}(\omega)}$. Suppose $X \subseteq Y$. Then E_X Hyp-reduces to E_Y via the map $G(x) = x$ if $x \in \bigcup_{i\in X} A_i$, $G(x) = (x(0), 0, 0, \ldots)$ for $x \notin \bigcup_{i\in X} A_i$.

Suppose $X \not\subseteq Y$ but $E_X \leq_H E_Y$ via a Hyp function F. Pick $i_0 \in X \setminus Y$. As before, we consider the set

$$A_{i_0} \cap F^{-1}(\sim \bigcup_{j\in Y} A_j).$$

Then this is a countable Hyp set. If it is non-empty then it contains a Hyp real, contradicting the definition of A_{i_0}. Therefore we get $F[A_{i_0}] \subseteq \bigcup_{j \in Y} A_j \subseteq \bigcup_{j \neq i_0} A_j$, contradiction. $\square$

Corollary 56. *There are infinite chains and antichains between $=_\omega$ and $=_{\mathcal{P}(\omega)}$.*

Corollary 57. *For any finite $n_0 \geq 1$, the partial order of Hyp subsets of ω under inclusion can be embedded into the structure of degrees of Hyp equivalence relations between $=_{n_0}$ and $=_{\mathcal{P}(\omega)}$ but incomparable with $=_n$ for $n_0 < n \leq \omega$.*

Proof. For every Hyp $X \subseteq \omega$, consider the equivalence relation of the form

$$\begin{aligned} xE_X^{n_0}y \iff & x \in \bigcup_{i \in X} A_i \wedge x = y \vee \\ & x, y \notin \bigcup_{i \in X} A_i \wedge (x(0) = y(0) < n_0 - 1 \vee x(0), y(0) \geq n_0 - 1). \end{aligned}$$

Then $E_X^{n_0}$ has exactly n_0 equivalence classes with Hyp reals. Therefore $=_{n_0} \leq_H E_X^{n_0}$ and for $n_0 < n \leq \omega$, the equivalence relation $=_n$ is incomparable with $E_X^{n_0}$. $\square$

Hyp Equivalence Relations between $=_{\mathcal{P}(\omega)}$ and E_0

Harrington-Kechris-Louveau showed that any Hyp equivalence relation is either Hyp reducible to $=_{\mathcal{P}(\omega)}$, or E_0 is Borel reducible to it. We now show that the latter option is not effective.

Theorem 58. *There exist Hyp-incomparable Hyp equivalence relations between $=_{\mathcal{P}(\omega)}$ and E_0.*

Proof sketch. Let A and B be Hyp sets such that for no Hyp function F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.

Define two Hyp equivalence relations E_A and E_B by

$$(x, y)E_A(x', y') \iff x = x' \wedge [(x \notin A) \vee (x \in A \wedge yE_0y')],$$

and similarly for E_B with B replacing A.

Suppose F is a Hyp-reduction of E_A to E_B. Define $F'(x,y) = z \iff (\exists w)F(x,y) = (z,w)$. Note that F' is constant on E_A classes. Define a function h by

$$\begin{aligned} h(x) = z &\iff \{y \in 2^\omega : g'(x,y) = z\} \text{ is non-meagre} \\ (&\iff \{y \in 2^\omega : g'(x,y) = z\} \text{ is comeagre.}) \end{aligned}$$

h is an everywhere defined Hyp function. Suppose $x \in A$. Then for a comeagre set C we have $F'(x,y) = h(x)$ for all $y \in C$. We claim that $h(x) \in B$. Indeed, otherwise the set $\{x\} \times C$ is mapped by g into a single E_B class, contradicting that all $E_A|\{x\} \times 2^\omega$ classes are meagre in $\{x\} \times 2^\omega$ (in fact, they are countable).

Thus h is a Hyp function with $h[A] \subseteq B$, contradicting the properties of A and B. □

Similarly we have:

Theorem 59. *The partial order of Hyp subsets of ω can be embedded into the structure of Hyp equivalence relations between $=_{\mathcal{P}(\omega)}$ and E_0.*

Theorem 60. *For any $n_0 \leq \omega$ the partial order of Hyp subsets of ω can be embedded into the structure of degrees of Hyp equivalence relations between $=_{n_0}$ and E_0, but incomparable with $=_n$ for $n_0 < n \leq \omega$ and incomparable with $=_{\mathcal{P}(\omega)}$.*

20.Vorlesung

Hyp Equivalence Relations with countably many classes

Proposition 61. *Let $1 \leq n \leq \omega$ and let E be a Hyp equivalence relation. Then $=_n \leq_H E$ iff E has at least n classes containing Hyp reals.*

Proof. ($\Rightarrow$): For every $1 \leq n \leq \omega$, the equivalence relation $=_n$ has exactly n equivalence classes and each of them contains a Hyp real. Under Hyp-reducibility Hyp reals are sent to Hyp reals and inequivalent reals are sent to inequivalent reals.
($\Leftarrow$): If n is finite, pick n Hyp reals $x_0, \ldots, x_{n-1}$ that lie in different equivalence classes of E. The function F that sends the i-th equivalence class of $=_n$ to x_i witnesses the reduction. Now suppose $n = \omega$. Suppose E is an equivalence relations with infinitely many classes containing Hyp reals. We

want to prove that $=_\omega$ Hyp-reduces to E. We will find a Hyp sequence of reals representing distinct equivalence classes of E. Consider the following relation $P(X,Y)$ on $\omega \times (\omega^\omega)^{<\omega}$:

$$P(X,Y) \iff [X = (n, X_0, \dots, X_n) \wedge \bigwedge_{i \neq j} \neg X_i E X_j] \longrightarrow$$
$$[Y = (n+1, Y_1, \dots, Y_n, Y_{n+1}) \wedge \bigwedge_i X_i = Y_i \wedge \bigwedge_{i \neq j} \neg Y_i E Y_j]$$

Then P is Hyp. Moreover, as E has infinitely many Hyp classes, for every Hyp X there exists a Hyp Y such that $P(X,Y)$. It follows from Hyp Dependent Choice that there exists a uniform sequence of Hyp sets $X_0, X_1, \dots$ such that

$$\forall i, j (i \neq j \to \neg X_i E X_j).$$

Then the function that sends the equivalence class $\{x : x(0) = n\}$ of $=_\omega$ to X_n is Hyp and witnesses the reduction. $\square$

Corollary 62. *If $=_n \leq_H E$, for all $1 \leq n < \omega$, then $=_\omega \leq_H E$.*

Proposition 63. *Let $1 \leq n \leq \omega$ and let E be a Hyp equivalence relation. Then $E \leq_H =_n$ iff E has at most n classes.*

Proof sketch. The direction ($\Rightarrow$) is obvious since non-equivalent reals are sent to non-equivalent reals under Hyp-reducibility. To prove ($\Leftarrow$) we need to show that the equivalence classes of a Hyp equivalence relation with at most countably many equivalence classes are uniformly Hyp.

By Harrington's proof of the Silver Dichotomy, if E has only countably many classes then every real belongs to a Hyp subset of some equivalence class. Let C be the set of codes for Hyp subsets of an equivalence class; then C is Π^1_1. Consider the relation

$$R = \{(x, c) \mid c \in C \text{ and } x \in H(c), \text{the Hyp set coded by } c\}.$$

Then R is Π^1_1 and can be uniformised by a Π^1_1 function F. As the values of F are numbers, F is Hyp and by separation we can choose a Hyp $D \subseteq C$, $D \supseteq \text{Range}\,(F)$. Now define an equivalence relation E^* on D by:

$$\begin{aligned} d_0 E^* d_1 &\iff (\forall x_0, x_1)((x_0 \in H(d_0) \wedge x_1 \in H(d_1)) \to x_0 E x_1) \\ &\iff (\exists x_0, x_1)(x_0 \in H(d_0) \wedge x_1 \in H(d_1) \wedge x_0 E x_1). \end{aligned}$$

i.e. $d_0 E^* d_1$ if and only if $H(d_0)$ and $H(d_1)$ are subsets of the same E-equivalence class. Note that E^* is Hyp. The relation E Hyp-reduces to E^* via $x \mapsto F(x)$. But E^* is just a Hyp relation on a Hyp set of numbers, so E^* is Hyp-reducible to $=_\omega$ (to see this, send c to the least number c^*, cE^*c^*).

Thus if E is a Hyp equivalence relation with at most countably many classes then E is Hyp-reducible to $=_\omega$. (In particular, all equivalence classes of E are Hyp.) One can similarly see that if E has at most n classes then E is Hyp-reducible to $=_n$. □

Incomparable Hyp equivalence relations

For the proofs of the next three results, we fix Π^0_1 sets A and B such that for no Hyp F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.

Theorem 64. *There are incomparable Hyp equivalence relations between $=_1$ and $=_2$.*

Proof. We take the equivalence relation E_A with two equivalence classes A, $\sim A$ and E_B with two equivalence classes B, $\sim B$. Then E_A and E_B are Hyp-reducible to $=_2$. By the properties of A and B, the relations E_A and E_B are Hyp-incomparable, as otherwise (since neither A nor B contain Hyp reals) we would have a Hyp function which maps A to B or vice versa. □

Theorem 65. *There exist Hyp-incomparable Hyp equivalence relations between $=_\omega$ and $=_{\mathcal{P}(\omega)}$.*

Proof. Consider the equivalence relations E_A and E_B:

$$xE_Ay \iff [(x \in A \wedge x = y) \vee (x, y \notin A \wedge x(0) = y(0))]$$

and similarly for E_B with B replacing A.

By sending the n-th class of $=_\omega$ to the real $(n, 0, 0, \dots)$ we get a Hyp reduction $=_\omega$ to E_A. Also E_A Hyp-reduces to $=_{\mathcal{P}(\omega)}$ via the map $G(x) = x$ if x belongs to A, $G(x) = (x(0), 0, 0, \dots)$ for $x \notin A$. Similarly for B.

There is no Hyp reduction of E_A to E_B. Indeed, suppose that F were such a reduction and let C be the preimage under F of $\sim B$. As $\sim B$ is Σ^0_1, C is Hyp and therefore $A \cap C$ is also Hyp. But $A \cap C$ must be countable as F is a reduction. So if $A \cap C$ were nonempty it would have a Hyp element, contradicting the fact that A has no Hyp elements. Therefore F maps A into B, which is impossible by the choice of A and B. □

Theorem 66. *There exist Hyp-incomparable Hyp equivalence relations between $=_{\mathcal{P}(\omega)}$ and E_0.*

Proof sketch. Define two Hyp equivalence relations E_A and E_B by

$$(x,y)E_A(x',y') \iff x = x' \wedge [(x \notin A) \vee (x \in A \wedge yE_0y')],$$

and similarly for E_B with B replacing A.

Suppose F is a Hyp-reduction of E_A to E_B. Define $F'(x,y) = z \iff (\exists w)F(x,y) = (z,w)$. Note that F' is constant on E_A classes. Define a function h by

$$\begin{aligned} h(x) = z &\iff \{y \in 2^\omega : F'(x,y) = z\} \text{ is non-meagre} \\ (&\iff \{y \in 2^\omega : F'(x,y) = z\} \text{ is comeagre.}) \end{aligned}$$

h is everywhere-defined (as a Baire-measurable function which is constant on E_0-classes is constant on a comeagre set). Also h is a Hyp function (as being meagre is a Π^1_1 property of the code for a Hyp set). Suppose $x \in A$. Then for a comeagre set C we have $F'(x,y) = h(x)$ for all $y \in C$. We claim that $h(x) \in B$. Indeed, otherwise the set $\{x\} \times C$ is mapped by F into a single E_B class, contradicting that all $E_A|\{x\} \times \omega^\omega$ classes are meagre in $\{x\} \times \omega^\omega$ (in fact, they are countable).

Thus h is a Hyp function with $h[A] \subseteq B$, contradicting the properties of A and B. $\square$