Absoluteness Course, Wintersemester 2004

Lectures 1 and 2

Introduction

This course will treat generalisations of the following classical result. For an infinite cardinal κ, $H(\kappa)$ denotes the set of sets whose transitive closure has cardinality less than κ.

Theorem 1. (Lévy Absoluteness) Suppose that φ is a Σ_1 formula with parameters from $H(\omega_1)$. If φ is true in some extension of V satisfying ZFC (for example, in a set-generic extension of V) then φ is true in V.

An elementary fact is that for any uncountable cardinal κ, $H(\kappa)$ is a Σ_1-elementary submodel of V. Therefore:

Corollary 2. Suppose that W is an extension of V satisfying ZFC. Then $H(\omega_1)^V$ is a Σ_1-elementary submodel of $H(\omega_1)^W$.

What follows is an outline of the generalisations of Corollary 2 that we will consider in this course. Many concepts will be mentioned in this outline that will only later be defined, when we prove the mentioned results.

Definition. Suppose that $\mathcal{P}$ is a definable class of posets. A *$\mathcal{P}$-generic extension* is a set-generic extension of V obtained by forcing with a poset in $\mathcal{P}$. Then *$\Sigma_n(H(\kappa))$-absoluteness for $\mathcal{P}$-forcing* means that $H(\kappa)^V$ is a Σ_n-elementary submodel of $H(\kappa)^W$ whenever W is a $\mathcal{P}$-generic extension of V. We abbreviate this as $\text{Abs}(\Sigma_n(H(\kappa)), \mathcal{P})$.

Thus Corollary 2 implies $\text{Abs}(\Sigma_1(H(\omega_1)), \text{set-forcing})$, i.e., $\text{Abs}(\Sigma_1(H(\omega_1)), \mathcal{P})$ where $\mathcal{P}$ = the class of all posets.

It follows that $\Sigma_2(H(\omega_1))$ formulas are persistent for set-generic extensions of V, in the sense that if such a formula holds in some set-generic extension then it holds in all larger set-generic extensions. Therefore it is reasonable to consider $\text{Abs}(\Sigma_2(H(\omega_1)), \mathcal{P})$ for various forcing notions $\mathcal{P}$.

Theorem 3. The following are equiconsistent:
1. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{set-forcing})$.

2. $\text{Abs}(\Sigma_2(H(\omega_1)), \omega_1\text{-preserving set-forcing})$.
3. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{stationary-preserving (at } \omega_1\text{) set-forcing})$.
4. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{proper set-forcing})$ and ω_1 is inaccessible to reals.
5. There is a reflecting cardinal, i.e., a regular cardinal κ such that $H(\kappa)$ is Σ_2-elementary in V.

Theorem 4. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{semiproper set-forcing})$ is consistent relative to ZFC.

Theorem 5. The following are equiconsistent:
1. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{ccc set-forcing})$ and ω_1 is inaccessible to reals.
2. There is a Schrittesser cardinal.

It is reasonable to consider $\text{Abs}(\Sigma_3(H(\omega_1)), \mathcal{P})$ provided one imposes the hypothesis that $\Sigma_3(H(\omega_1))$ formulas persist for $\mathcal{P}$-generic extensions. The latter is equivalent to saying that $\text{Abs}(\Sigma_2(H(\omega_1)), \mathcal{P})$ holds in all $\mathcal{P}$-generic extensions, a form of "two-step absoluteness" for $\mathcal{P}$-forcing. We consider next some examples of this.

Theorem 6. The following are equivalent:
1. All set-generic extensions obey $\Sigma_2(H(\omega_1))$-absoluteness for further set-generic extensions.
2. All stationary-preserving at ω_1 set-generic extensions obey $\Sigma_2(H(\omega_1))$-absoluteness for further stationary-preserving at ω_1 set-generic extensions.
3. Every set has a #.

Theorem 7? The following are equivalent:
1. All proper set-generic extensions obey $\Sigma_2(H(\omega_1))$-absoluteness for further proper set-generic extensions.
2. Every set belongs to an inner model with a remarkable cardinal.

Theorem 8? The following are equivalent:
1. All ccc set-generic extensions obey $\Sigma_2(H(\omega_1))$-absoluteness for further ccc set-generic extensions.
2. ω_1 is weakly compact relative to reals.

There should be a version of Theorems 6-8 for semiproper forcing.

In light of Theorem 6, the correct context for $\Sigma_3(H(\omega_1))$-absoluteness for set-generic extensions is ZFC + Every set has a #.

Theorem 9. The following are equiconsistent:
1. $\Sigma_3(H(\omega_1))$-absoluteness for set-generic extensions and every set has a #.
2. There exists a reflecting cardinal and every set has a #.

There should be results analogous to Theorem 9 for semiproper, proper and ccc.

$\Sigma_4(H(\omega_1))$-absoluteness for set-generic extensions is reasonable provided $\Sigma_4(H(\omega_1))$ formulas persist for set-generic extensions, i.e., provided that all set-generic extensions obey $\Sigma_3(H(\omega_1))$-absoluteness for further set-generic extensions.

Theorem 10? The following are equivalent:
1. All set-generic extensions obey $\Sigma_3(H(\omega_1))$-absoluteness for further set-generic extensions.
2. Every set belong to an inner model with a strong cardinal.

Theorem 11. The following are equiconsistent:
1. $\Sigma_4(H(\omega_1))$-absoluteness and every set belongs to an inner model with a strong cardinal.
2. There exists a reflecting cardinal and every set belongs to an inner model with a strong cardinal.

To continue, one adds strong cardinals.

There should be appropriate versions of Theorems 10, 11 for semiproper, proper and ccc.

We next consider absoluteness for $H(\omega_2)$. This is particularly interesting due to its connections to the "bounded forcing axioms".

Theorem 12. $\text{Abs}(\Sigma_1(H(\omega_2)), \omega_1\text{-preserving set-forcing})$ is false.

Theorem 13. $\text{Abs}(\Sigma_1(H(\omega_2)), \text{ccc set-forcing})$ is equivalent to Martin's Axiom at ω_1.

Theorem 14. The following are equiconsistent:
1. $\text{Abs}(\Sigma_1(H(\omega_2)), \text{proper set-forcing})$.
2. $\text{Abs}(\Sigma_1(H(\omega_2)), \text{semiproper set-forcing})$.
3. There is a reflecting cardinal

Theorem 15. Abs($\Sigma_1(H(\omega_2))$, stationary-preserving at ω_1 set-forcing) implies that every set belongs to an inner model with a strong cardinal. The consistency of Abs($\Sigma_1(H(\omega_2))$, stationary-preserving at ω_1 set-forcing) follows from that of a proper class of Woodin cardinals.

Theorem 16. $\Sigma_1(H(\omega_2))$-absoluteness cannot hold in all ccc set-forcing extensions.

So it is not reasonable to look at $\Sigma_2(H(\omega_2))$-absoluteness.

Theorem 17. $\Sigma_1(H(\omega_3))$-absoluteness for ccc set-forcing extensions is equivalent to Martin's Axiom at ω_2. But for proper set-forcing extensions it is false. Also, for set-forcing extensions which are stationary-preserving at both ω_1 and ω_2 it is false?

An appropriate form of $\Sigma_1(H(\omega_3))$-absoluteness for more than ccc forcing is not known.

Absoluteness principles on $H(\omega_1)$ have no effect on the size of the continuum. However those on $H(\omega_2)$ do:

Theorem 18. $\Sigma_1(H(\omega_2))$-absoluteness for proper set-forcing implies $2^{\aleph_0} = \aleph_2$.

Strong Absoluteness

Notice that Lévy absoluteness applies to arbitrary extensions, not just set-generic ones. Are there strengthenings of Lévy absoluteness which also apply to arbitrary extensions?

Theorem 19. $\Sigma_2(H(\omega_1))$-absoluteness (and hence also $\Sigma_1(H(\omega_2))$-absoluteness) for (stationary-preserving at ω_1) class-forcing extensions is false.

A consistent possibility is to require *absolute parameters.* A class A is *absolute between V and an extension W* iff some formula without parameters defines A both in V and in W.

Conjecture. The following axiom is consistent relative to large cardinals:

Strong Absoluteness. Absoluteness holds for arbitrary extensions of V for Σ_1 formulas with absolute class parameters: If a Σ_1 formula φ with class parameter A holds in an extension W of V and A is absolute between V and W then φ holds in V.

Theorem 20? $(*)$ implies the existence of an inner model with a Woodin cardinal.

Lectures 3 and 4

Definitions and Proofs

We now begin the formal part of the course. Our first task is to prove:

Theorem 1. (Lévy Absoluteness) Suppose that φ is a Σ_1 formula with parameters from $H(\omega_1)$. Suppose that W is an *outer model* of V (i.e., an extension of V satisfying ZFC with same ordinals as V; for example, a set-generic extension of V). Then if φ is true in W it is also true in V.

Proof. The idea is to associate to each Σ_1 formula φ with parameter $x \in H(\omega_1)$ a tree T_φ of size ω_1^W such that in both V and W, φ is true iff T_φ has an infinite branch (i.e., iff T_φ is not well-founded). This reduces Lévy absoluteness to the absoluteness of well-foundedness of trees, a simple consequence of the ZFC axioms.

To obtain the tree T_φ we proceed as follows. For simplicity, assume that x does not exist, i.e., that φ has no parameter. (The proof we give will "relativise" to the parameter x, so this is not a serious restriction.) As φ is Σ_1 it is equivalent to

There exists a transitive set t such that $t \vDash \varphi$.

In fact, φ is equivalent to

There exists a countable transitive set t such that $t \vDash \varphi$,

since if u is an arbitrary transitive set satisfying φ, we can replace u by the transitive collapse of a countable elementary submodel of u, which will then be a countable transitive model of φ.

Now for each countable transitive set t, the structure $(t, \in)$ is isomorphic to a structure (ω, E) where E is a binary relation on ω. Conversely, by the Mostowski Collapse Theorem, if (ω, E) satisfies the Axiom of Extensionality and is well-founded, then it is isomorphic to $(t, \in)$ for some countable transitive t. Therefore φ is equivalent to

There exists an (ω, E) satisfying both φ and the Axiom of Extensionsality which is well-founded.

Let ψ be the conjunction of φ with the Axiom of Extensionality. Write ψ in prenex normal form, for example, as $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \gamma(x_1, \ldots, x_4)$, where γ is quantifier-free. Then (ω, E) satisfies ψ iff there exist Skolem functions $f_1 : \omega \to \omega$, $f_{1,3} : \omega \times \omega \to \omega$ such that $(\omega, E) \vDash \forall x_1 \forall x_3 \gamma(x_1, f_1(x_1), x_3, f_{1,3}(x_1, x_3))$, the latter being a universal formula.

We now describe a tree T' with the property that T' has an infinite branch iff some (ω, E), possibly ill-founded, satisfies both φ and the Axiom of Extensionality. A node (element) of T' on level n is a finite structure (s, e) where e is a binary relation on s and s is a finite set of natural numbers containing $n = \{0, 1, \ldots, n-1\}$, together with functions $f_1^s : n \to s$, $f_{1,3}^s : n \times n \to s$ such that $s = \text{Ran } f_1^s \cup \text{Ran } f_{1,3}^s$ and for all $x_1, x_3 < n$, $(s, e) \vDash \gamma(x_1, f_1^s(x_1), x_3, f_{1,3}^s(x_1, x_3))$. When extending a node, one increases n, enlarges the structure (s, e) and extends the functions $f_1^s, f_{1,3}^s$. Then an infinite branch through this tree produces a model (ω, E) of ψ, i.e., of φ together with the Axiom of Extensionality. Conversely, if ψ has a model then this tree will have an infinite branch.

We need to modify T' to a tree T whose infinite branches correspond to *well-founded* models (ω, E) of ψ. A node of T consists of (s, e) and $f_1^s, f_{1,3}^s$ as above, together with a function $r : s \to \omega_1^W$ with the property that if the pair (m, n) belongs to e, then $r(m) < r(n)$. Then an infinite branch through T gives rise to a model (ω, E) of ψ together with a "ranking function" $R : \omega \to \omega_1$ with the property that $(m, n) \in E$ implies $R(m) < R(n)$; it follows that the model (ω, E) must be well-founded. Conversely, if ψ has a countable well-founded model in W then the tree T will have an infinite branch, since we can choose a ranking function for that model with values less than ω_1^W.

So the truth of φ is equivalent to the existence of an infinite branch through T, and this equivalence holds not only in V, but also in W. If T has an infinite branch in V then of course it also has an infinite branch in W, since W contains V. Conversely, suppose that T has no infinite branch in V. Then since V satisfies ZFC, in V there is a “ranking function” G on T, i.e., a function G from the nodes of T into Ord such that if a is a node of T extending the node b of T, then $G(a) < G(b)$. As the function G also belongs to W, it follows that T has no infinite branch in W. Therefore we have:

φ is true in V iff
T has an infinite branch in V iff
T has an infinite branch in W iff
φ is true in W,

as desired. $\Box$

Corollary 2. Suppose that W is an extension of V satsifying ZFC. Then $H(\omega_1)^V$ is a Σ_1-elementary submodel of $H(\omega_1)^W$.

Proof. Suppose that a Σ_1 formula with parameters from $H(\omega_1)^V$ is true in $H(\omega_1)^W$. Then it is also true in W and therefore by Lévy absoluteness, in V. Therefore we need only show that $H(\omega_1)^V$ is Σ_1-elementary in V. But if a Σ_1 formula with parameters from $H(\omega_1)^V$ is true in V, it is also true in a countable Σ_1-elementary submodel M of V, in the transitive collapse T of M and therefore also in $H(\omega_1)^V$, since T is a transitive submodel of $H(\omega_1)^V$. $\Box$

It follows that $\Sigma_2(H(\omega_1))$ formulas are *persistent*, in the sense that if $V \subseteq W$ are models of ZFC with the same ordinals then any $\Sigma_2(H(\omega_1))$ formula true in V is also true in W. For this reason, it is natural to consider $\text{Abs}(\Sigma_2(H(\omega_1), \mathcal{P})$, for various set-forcing notions $\mathcal{P}$, our next topic.

We first prove

Theorem 3.1. The following are equiconsistent:
1. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{set-forcing})$.
2. There exists a *reflecting* cardinal, i.e., a regular cardinal κ such that $H(\kappa)$ is Σ_2-elementary in V.

Proof. First suppose that κ is Σ_2-reflecting, and we show that $V[G]$ satisfies $\text{Abs}(\Sigma_2(H(\omega_1)), \text{set-forcing})$, where G is generic over V for $\text{Coll}(\omega, < \kappa)$, the forcing that with finite conditions collapses every ordinal less than κ to ω. A condition in $\text{Coll}(\omega, < \kappa)$ is a function p with domain a finite subset of

$\omega \times \kappa$ such that $p(n, \alpha) < \alpha$ for each $(n, \alpha) \in \mathrm{Dom}(p)$. For any $\bar{\kappa} < \kappa$ let $\mathrm{Coll}(\omega, < \bar{\kappa})$ denote the set of conditions in $\mathrm{Coll}(\omega, < \kappa)$ with domain contained in $\omega \times \bar{\kappa}$.

Lemma 3.1.1. (a) Suppose that $\bar{\kappa}$ is a limit ordinal less than κ and $\bar{X}$ is a maximal antichain in $\mathrm{Coll}(\omega, < \bar{\kappa})$. Then $\bar{X}$ is a maximal antichain in $\mathrm{Coll}(\omega, < \kappa)$.
(b) $\mathrm{Coll}(\omega, < \kappa)$ has the κ-cc, i.e., antichains in this forcing have size less than κ.
(c) If G is $\mathrm{Coll}(\omega, < \kappa)$-generic then $G \cap \mathrm{Coll}(\omega, < \bar{\kappa})$ is $\mathrm{Coll}(\omega, < \bar{\kappa})$-generic for each limit ordinal $\bar{\kappa} < \kappa$.
(d) If x is a real in $V[G]$, where G is $\mathrm{Coll}(\omega, < \kappa)$-generic over V, then x belongs to $V[G \cap \mathrm{Coll}(\omega, < \bar{\kappa})]$ for some limit $\bar{\kappa} < \kappa$.

Proof. (a) It suffices to show that every condition in $\mathrm{Coll}(\omega, < \kappa)$ is compatible with some element of $\bar{X}$. Given a condition p, let $\bar{p}$ be p restricted to $\mathrm{Dom}(p) \cap (\omega \times \bar{\kappa})$. Then $\bar{p}$ belongs to $\mathrm{Coll}(\omega, < \bar{\kappa})$ and therefore is compatible with some $\bar{q}$ in $\bar{X}$. But then p is also compatible with $\bar{q}$, since $\bar{q}$ and $p - \bar{p}$ have disjoint domains.
(b) This follows from (a), since if X is a maximal antichain in $\mathrm{Coll}(\omega, < \kappa)$, a closure argument shows that $\bar{X} = X \cap \mathrm{Coll}(\omega, < \bar{\kappa})$ is a maximal antichain in $\mathrm{Coll}(\omega, < \bar{\kappa})$ for some limit $\bar{\kappa} < \kappa$.
(c) This also follows from (a).
(d) Let $x = \sigma^G$ (i.e., σ is a name for x in $V[G]$). For each n, the set of conditions in $\mathrm{Coll}(\omega, < \kappa)$ which decide the sentence "$n \in \sigma$" is dense. Note that any maximal antichain in a dense subordering of P is also a maximal antichain in P. For each n let X_n be a maximal antichain of conditions which decide "$n \in \sigma$". Then each X_n is a maximal antichain in P and x is determined by how G intersects the X_n's. By (b), there is a limit $\bar{\kappa} < \kappa$ such that each X_n is contained in $\mathrm{Coll}(\omega, < \bar{\kappa})$, and therefore x belongs to $V[G \cap \mathrm{Coll}(\omega, < \bar{\kappa})]$.
$\square$ (Lemma 3.1.1)

Now suppose that φ is a $\Sigma_2(H(\omega_1))$ formula with parameter $p \in H(\omega_1)^{V[G]}$ which is true in some set-generic extension of $V[G]$. Note that any $\Sigma_2(H(\omega_1))$ formula is also a Σ_2 formula, as the relation "$x \in H(\omega_1)$" is Σ_1. First assume that p belongs to V; since the transitive closure of p is countable in $V[G]$, p in fact belongs to $H(\kappa)^V$. Then V satisfies the following sentence with parameter p:

There exists a set-forcing P such that $P \Vdash \varphi$.

As φ is Σ_2, the relation "$P \Vdash \varphi$" is also Σ_2. As κ is reflecting, the above sentence is true in $H(\kappa)$. Let P be a set-forcing in $H(\kappa)$ which forces φ. Then since κ is strongly inaccessible, the power set of P has size less than κ in V and therefore is countable in $V[G]$. It follows that in $V[G]$ there exists a P-generic g over V, and since P forces φ, we get $V[g] \vDash \varphi$. Since φ is $\Sigma_2(H(\omega_1))$, by persistence it is also true in $V[G]$, as desired.

If p is not 0, then we argue as follows. As p belongs to $H(\omega_1)^{V[G]}$, it can be coded by a real in $V[G]$. By Lemma 3.1.1 (d), p belongs to $V[G \cap \mathrm{Coll}(\omega, <\bar{\kappa})]$ for some limit $\bar{\kappa} < \kappa$. Now the forcing $\mathrm{Coll}(\omega, <\kappa)$ factors as $\mathrm{Coll}(<\bar{\kappa}) \times \mathrm{Coll}(\geq \bar{\kappa})$, and therefore $V[G] = V[G(<\bar{\kappa})][G(\geq \bar{\kappa})]$, where $G(\geq \bar{\kappa})$ is $P(\geq \bar{\kappa})$-generic over $V[G(<\bar{\kappa})$. Now repeat the above argument using the ground model $V[G(<\bar{\kappa})$, which contains the parameter p, and its $P(\geq \bar{\kappa})$-generic extension $V[G(<\bar{\kappa})][G(\geq \bar{\kappa})] = V[G]$.

Lectures 5 and 6

We complete the proof of

Theorem 3.1. The following are equiconsistent:
1. $\mathrm{Abs}(\Sigma_2(H(\omega_1)), \text{set-forcing})$.
2. There exists a *reflecting* cardinal, i.e., a regular cardinal κ such that $H(\kappa)$ is Σ_2-elementary in V.

Proof. It remains to show that the consistency of 1 implies that of 2. We show that if $\mathrm{Abs}(\Sigma_2(H(\omega_1)), \text{set-forcing})$ holds, then $\kappa = \omega_1^V$ is a reflecting cardinal in L. Suppose that $L \vDash \varphi$, where φ is a Σ_2 formula with parameters from $H(\kappa)^L = L_\kappa$. We must show that φ is true in L_κ. Since φ is true in L, by reflection it is also true in $H(\theta)^L = L_\theta$ for some L-cardinal θ. There is a set-generic extension of V in which θ is countable. Therefore in some set-generic extension of V the following $\Sigma_2(H(\omega_1))$ formula (with parameters from $L_\kappa \subseteq H(\omega_1)^V$) is true:

$H(\omega_1) \vDash$ There exists an ordinal θ such that θ is a cardinal of L and $L_\theta \vDash \varphi$.

By $\mathrm{Abs}(\Sigma_2(H(\omega_1)), \text{set-forcing})$, the above formula is also true in V. Therefore there is an ordinal θ less than $\omega_1^V = \kappa$ such that $L_\theta \vDash \varphi$ and $H(\omega_1)^V \vDash \theta$ is an L-cardinal. Since $H(\omega_1)^V$ is Σ_1-elementary in V, θ really is an L-cardinal, and therefore L_θ is Σ_1-elementary in L_κ. As φ is Σ_2, it follows that L_κ also satisfies φ, as desired. $\square$

Iterated Set-Forcing and Properness

We now consider $\mathrm{Abs}(\Sigma_2(H(\omega_1)), \mathcal{P})$ for various types of set-forcing $\mathcal{P}$. Some natural choices for $\mathcal{P}$ are the following

ccc $\subseteq$ Proper $\subseteq$ Stationary-preserving at $\omega_1 \subseteq \omega_1$-preserving $\subseteq$ Set-forcing

A forcing P is *stationary-preserving at* ω_1 iff whenever $X \subseteq \omega_1$ is stationary, it remains stationary in each P-generic extension. The definition of *proper* is more complex, and is closely related to the method of *forcing iteration*, which we describe next.

First we consider *finite-support iteration.*

Definition. Let α be a nonzero ordinal. P_α is an *iteration of length* α *with finite support* iff it is a set of α-sequences with the following properties:
(i) If $\alpha = 1$ then for some forcing notion $Q_0 = \dot{Q}_0$, P_1 is the set of all sequences $\langle p(0) \rangle$ of length 1, where $p(0) \in Q_0$. And $\langle p(0) \rangle \leq \langle q(0) \rangle$ iff $p(0) \leq q(0)$, $1^{P_1} = \langle 1^{Q_0} \rangle$.
(ii) If $\alpha = \beta + 1$ then $P_\beta = \{p \restriction \beta \mid p \in P_\alpha\}$ is an iteration of length β and there is some P_β-name $\dot{Q}_\beta$ such that $1^{P_\beta} \Vdash \dot{Q}_\beta$ is a forcing notion and:
$p \in P_\alpha$ iff $p \restriction \beta \in P_\beta$, $p(\beta)$ is a P_β-name of rank less than Rank $\dot{Q}_\beta$ and $1^{P_\beta} \Vdash p(\beta) \in \dot{Q}_\beta$.
$p \leq q$ in P_α iff $p \restriction \beta \leq q \restriction \beta$ in P_β and $p \restriction \beta \Vdash p(\beta) \leq q(\beta)$, and 1^{P_α} is defined by $1^{P_\alpha}(\gamma) = 1^{\dot{Q}_\gamma}$ for all $\gamma \leq \beta$.
(iii) If α is a limit ordinal then for all $\beta < \alpha$, $P_\beta = \{p \restriction \beta \mid p \in P_\alpha\}$ is an iteration of length β and:
$p \in P_\alpha$ iff
$p \restriction \beta \in P_\beta$ for all $\beta < \alpha$ and $1^{P_\beta} \Vdash p(\beta) = 1^{\dot{Q}_\beta}$ for all but finitely many $\beta < \alpha$.
Also: $p \leq q$ in P_α iff $p \restriction \beta \leq q \restriction \beta$ in P_β for all $\beta < \alpha$ and 1^{P_α} is defined by $1^{P_\alpha}(\beta) = 1^{\dot{Q}_\beta}$ for all $\beta < \alpha$.

Notation. $\leq_\beta$ denotes the ordering of P_β, $\Vdash_\beta$ denotes the forcing relation of P_β and $\Vdash_\beta \varphi$ abbreviates $1^{P_\beta} \Vdash_\beta \varphi$.

Theorem 3.2.1. Let P_α result from the iteration with finite support of $\langle \dot{Q}_\beta \mid \beta < \alpha \rangle$. If $\Vdash_\beta \dot{Q}_\beta$ is ccc for each $\beta < \alpha$ then P_α is ccc.

A nice application of finite support iteration is to Suslin's Problem. Suslin asked whether there is a complete, dense linear ordering without endpoints, without an uncountable set of pairwise disjoint intervals and not isomorphic to the real line. It turned out the answer is Yes in L, but the answer is No in an extension of L obtainable through iteration with finite support.

An equivalent version of Suslin's question is the following: Is there a Suslin Tree? The latter is an uncountable partially-ordered set $(T, <_T)$ such that the predecessors of each element of T are well-ordered by $<_T$ and $(T, <_T)$ has no uncountable chain or antichain.

Notice that if $(T, <_T)$ is a Suslin tree then $(T, \geq_T)$ is a partial-ordering and therefore can be used as a forcing notion. If T is a Suslin tree with the property that each $t \in T$ has uncountably many extensions in T, then forcing with T adds an $\aleph_1$-branch through T and therefore T will not be Suslin in the generic extension.

Theorem 3.2.2. In L, there is an iteration with finite support P of length $\aleph_2$ such that if G is P-generic over L then in $L[G]$ there are no Suslin trees.

Iterations with countable support are defined just like iterations with finite support, but with the condition at limit stages α given as follows:

$p \in P_\alpha$ iff $p \restriction \beta \in P_\beta$ for all $\beta < \alpha$ and
$1^{P_\beta} \Vdash p(\beta) = 1^{\dot{Q}_\beta}$ for all but *countably* many $\beta < \alpha$.

This type of iteration is needed when one wishes to use forcings which are not ccc. Often one performs an iteration of length $\aleph_2$, using forcings of size $\aleph_1$. To show that cardinals above $\aleph_1$ are preserved one uses:

Proposition 3.2.3. Let P be a countable support iteration of length $\aleph_2$ such that for $\beta < \aleph_2$, $P \restriction \beta$ has the $\aleph_2$-cc. Then P has the $\aleph_2$-cc.

How does one show that $\aleph_1$ is preserved in a countable support iteration? One way is to assume that the forcings used are countably closed (i.e., every countable descending sequence of conditions has a lower bound). However this is too restrictive for applications. Shelah isolated a useful condition on the forcings used in the iteration, called *properness*, which guarantees preservation of $\aleph_1$, is maintained through countable support iteration and has many applications.

Definition. P is *proper* iff player II has a winning strategy in the following game: Player I begins by selecting a condition p and choosing a name $\dot{\alpha}_0$ for an ordinal. Player II chooses an ordinal β_0. At the n-th move, $n > 0$, I plays a name $\dot{\alpha}_n$ for an ordinal and II plays an ordinal β_n. Now II wins the game iff for some $q \leq p$:

$$(*)\ q \Vdash \text{For all } n,\ \dot{\alpha}_n \text{ equals } \beta_k \text{ for some } k.$$

Notice that if II has a winning strategy in the above game, then every countable set of ordinals in a P-generic extension of V is a subset of a set of ordinals which is countable in V. Thus properness implies that $\aleph_1$ is preserved. It is not difficult to show that the same definition of properness reuslts if we modify the above game so as to allow player II to play countable sets of ordinals rather than single ordinals (where II wins iff some $q \leq p$ forces that each ordinal name played by I belongs to the union of the countable sets of ordinals played by II).

Proposition 3.2.4. The following are equivalent:
1. P is proper.
2. For any uncountable κ, every stationary $A \subseteq P_{\omega_1}(\kappa)$ remains stationary after forcing with P.
3. For κ greater than the cardinality of the power set of P, there are CUB-many countable $M \prec H(\kappa)$ such that any $p \in M$ can be extended to $q \in P$ which is (P, M)-*generic*: If $D \in M$ is dense on P then q forces the generic to intersect $D \cap M$.

It is easy to see that any ccc forcing and any countably closed forcing is proper.

Theorem 3.2.5. Let P_γ be a countable support iteration of length γ of $\dot{Q}_\beta$, $\beta < \gamma$ such that for every $\beta < \gamma$, $\Vdash_\beta \dot{Q}_\beta$ is proper. Then P_γ is proper.

A nice application of countable support iteration is to prove the consistency of the Borel Conjecture. Let X be a subset of $[0, 1]$. X has *strong* *measure* 0 iff for every sequence $\langle \epsilon_n \mid n \in \omega \rangle$ of positive reals there exists a sequence $\langle I_n \mid n \in \omega \rangle$ of intervals with length $I_n \leq \epsilon_n$ such that $X \subseteq \bigcup_n I_n$. Borel conjectured that strong measure 0 sets are in fact countable. This contradicts CH, but Laver proved the consistency of Borel's Conjecture using a countable support iteration of *Laver forcing*.

Laver forcing is defined as follows. A set $p \subseteq \omega^{<\omega}$ is a *tree* iff it is closed under initial segments. A tree p is a *Laver tree* iff for some $s \in p$ (called the *stem* of p):

1. For all $t \in p$ either $t \subseteq s$ or $s \subseteq t$.
2. For all $t \in p$ extending s the set $S(t) = \{a \mid t * a \in p\}$ (the set of successors of t in p) is infinite.

Laver forcing consists of all Laver trees, partially ordered by inclusion. If G is generic then $f = \bigcup\{s \mid s \text{ is the stem of some } p \in G\}$ is a function from ω into ω, a *Laver real*. Laver forcing is neither ccc nor countably closed.

By Proposition 3.2.3, if we iterate Laver forcing with countable support for $\aleph_2$ steps over L, we will have the $\aleph_2$-cc and therefore preserve all cardinals greater than $\aleph_1$. To show that this iteration preserves $\aleph_1$, it suffices to show

Lemma 3.2.6. Laver forcing is proper.

Proof. Define the relations $\leq_n$ as follows. Consider a canonical enumeration of $\omega^{<\omega}$ in which s appears before t when $s \subseteq t$ and in which $s * a$ appears before $s*(a+1)$ for $a \in \omega$. If p is a Laver tree then the part of p above the stem is isomorphic to $\omega^{<\omega}$ and so we have a canonical enumeration $\langle s_i^p \mid i \in \omega \rangle$ of it, where s_0^p is the stem of p. Note that if $q \leq p$ and $s_n^q = s_m^p$ then $n \leq m$. We define:

$q \leq_n p$ iff p and q have the same stem and $s_i^p = s_i^q$ for all $i \leq n$.

It is easy to show that if $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$ then $p = \bigcap_n p_n$ is a Laver tree, called the *fusion of the fusion sequence* $\langle p_n \mid n \in \omega \rangle$.

Fact. If $p \Vdash \dot{\alpha} \in \mathrm{Ord}$, $m \in \omega$ then there are $q \leq_m p$ and a countable $A \subseteq \mathrm{Ord}$ such that $q \Vdash \dot{\alpha} \in A$.

Proof of Fact. We assume that $m = 0$, as the proof for general m is almost the same. If p is a Laver tree, $n \in \omega$, $q \leq p$ and the stem t of q is maximal among $\{s_0^p, \dots, s_n^p\}$ then

$$r = q \cup \{u \in p \mid u \not\subseteq t \text{ and } t \not\subseteq u\}$$

is a Laver tree $\leq_n p$, called *the n-amalgamation of q into p*. This has the obvious generalisation to *the amalgamation of $\{q_1, \dots, q_k\}$ into p* when the q_i extend p and their stems are all the maximal nodes among $\{s_0^p, \dots, s_n^p\}$ (for a uniquely determined n).

We construct a fusion sequence $\langle p_n \mid n \in \omega \rangle$ with $p_0 = p$ and finite sets A_n so that the fusion of this sequence forces $\dot\alpha \in \bigcup_n A_n$. At stage n we already have p_n. Let $t_1, \dots, t_k$ be all the maximal nodes among $s_0^{p_n}, \dots, s_n^{p_n}$. For each $i \in \{1, \dots, k\}$ if there exists $q_i \leq p_n$ with stem t_i and an ordinal α_n^i so that $q_i \Vdash \dot\alpha = \alpha_n^i$ then we choose such q_i and α_n^i. Let A_n be the collection of all the α_n^i chosen and let p_{n+1} be the amalgamation of $\{q_1, \dots, q_k\}$ into p_n. (If q_i did not exist, then we take it to be the collection of nodes in p_n compatible with t_i.) We have $p_{n+1} \leq_n p_n$.

Let p_∞ be the fusion of the p_n's and $A = \bigcup_n A_n$. To prove that $p_\infty \Vdash \dot\alpha \in A$, let $q \leq p_\infty$. There are $\bar q \leq q$ and $\alpha \in \mathrm{Ord}$ such that $\bar q \Vdash \dot\alpha = \alpha$. Let n be large enough so that the stem of $\bar q$ is among $K = \{s_0^{p_n}, \dots, s_n^{p_n}\}$. There is $t \in \bar q$ that is a maximal node in K and therefore one of the nodes considered at stage n, say $t = t_i$. Let r consist of those nodes of $\bar q$ which are compatible with t. As r and α satisfy the requirements for choosing q_i in the definition of p_{n+1} we indeed have chosen q_i and α_n^i. Because $r \leq q_i$ it must be the case that $\alpha = \alpha_n^i$ and so $r \Vdash \dot\alpha \in A$. Thus each $q \leq p_\infty$ has an extension r such that $r \Vdash \dot\alpha \in A$. Therefore $p_\infty \Vdash \dot\alpha \in A$. This proves the Fact.

Now we can show that II wins the proper game for Laver forcing (in the version where I plays a condition p and names for single ordinals, II plays countable sets of ordinals and II wins iff there is $q \leq p$ which forces all the names to be in the union of the sets or ordinals played). At the start of the game let I select p_0 and the ordinal name $\dot\alpha_0$. By the Fact there is $p_1 \leq_0 p_0$ and a countable B_0 such that $p_1 \Vdash \dot\alpha \in B_0$. At the nth move, when I plays $\dot\alpha_n$ there are $p_{n+1} \leq_n p_n$ and a countable set B_n with $p_{n+1} \Vdash \dot\alpha_n \in B_n$. Then the fusion of the p_n's verifies that II wins the game. □

Laver proves the consistency of Borel's Conjecture by showing: If GCH holds in V and X is an uncountable set of reals in V then X does not have strong measure 0 in $V[G]$ where G is generic over V for the countable support $\aleph_2$-iteration of Laver forcing.

Lectures 7 and 8

We now return to the study of absoluteness.

Theorem 3.2. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{Proper})$ is consistent relative to ZFC.

Proof. By a proper ω_1-iteration with countable support $\langle P_i \mid i < \omega_1 \rangle$, we can produce a generic extension $L[\langle G_i \mid i < \omega_1 \rangle]$ of L which satisfies absoluteness for $\Sigma_2(H(\omega_1))$ formulas with parameters from L with respect to further proper set-forcing extensions. This is possible as there are only ω_1 reals in L and properness is preserved by countable support iteration. We can further guarantee that for each $i < \omega_1$, $L[G_i] = L[X_i]$ for some $X_i \subseteq \omega_1$: At stage i, first force to guarantee absoluteness for some formula with a constructible parameter, and then force with the countably-closed (and therefore proper) forcing that collapses the cardinality of this forcing to ω_1 using countable conditions. The result is a model of the form $L[\langle X_i \mid i < \omega_1 \rangle]$ with $X_i \subseteq \omega_1$ for each i, satisfying absoluteness for $\Sigma_2(H(\omega_1))$ formulas with parameters from L with respect to further proper extensions. By dove-tailing, we can in fact ensure absoluteness for $\Sigma_2(H(\omega_1))$ formulas with parameters from $\bigcup_{i<\omega_1} L[G_i]$ with respect to further proper extensions.

Claim. Every real in $L[\langle X_i \mid i < \omega_1 \rangle]$ belongs to $L[\langle X_i \mid i < j \rangle]$ for some $j < \omega_1$.

Proof of Claim: If R is a real in $L[\langle X_i \mid i < \omega_1 \rangle]$ then R belongs to a countable, sufficiently elementary submodel M of $L[\langle X_i \mid i < \omega_1 \rangle]$, as well as to the transitive collapse $\bar{M}$ of M. But $\bar{M}$ is of the form $L_\alpha[\langle X_i \cap \beta \mid i < \beta \rangle]$ where β is the ω_1 of $\bar{M}$. It follows that R belongs to $L[\langle X_i \mid i < \beta \rangle]$, proving the Claim.

Thus $L[\langle X_i \mid i < \omega_1 \rangle]$ is a model of $\text{Abs}(\Sigma_2(H(\omega_1)), \text{Proper})$, as desired. □

Notice that in the model of the previous result, ω_1 is the same as ω_1^L. The next result implies that if $\omega_1^{L[R]}$ is collapsed for each real R, then $\Sigma_2(H(\omega_1))$ absoluteness for proper set-forcing extensions is as strong (in terms of consistency) as for arbitrary set-forcing extensions.

Theorem 3.3. The following are equiconsistent:
1. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{Proper})$ holds and ω_1 is *inaccessible to reals* (i.e., $\omega_1^{L[R]}$ is countable for each real R).
2. There exists a reflecting cardinal.

Proof. The consistency of 2 implies that of 1, as by Theorem 3.1 it even implies the consistency of $\Sigma_2(H(\omega_1))$-absoluteness for arbitrary set-forcings.

For the converse we shall need some facts about $0^\#$. The existence of $0^\#$ is equivalent to the statement that the uncountable cardinals form a class of order indiscernibles in L: For any formula $\varphi(x_1, \ldots, x_n)$, L satisfies $\varphi(\kappa_1, \ldots, \kappa_n)$ for *some* increasing n-tuple $\kappa_1 < \cdots < \kappa_n$ of uncountable cardinals iff L satisfies $\varphi(\kappa_1, \ldots, \kappa_n)$ for *all* such increasing n-tuples. This implies that all uncountable cardinals are reflecting, Mahlo and much more in L. The existence of $0^\#$ can also be characterised in terms of a relationship between the cardinals of V and those of L:

Theorem 3.3.1. (a) Suppose that $0^\#$ exists. Then for every cardinal κ, κ^+ of L is less than κ^+. (b) Conversely, if κ^+ of L is less than κ^+ for some singular cardinal κ, then $0^\#$ exists.

Assume now $\text{Abs}(\Sigma_2(H(\omega_1)), \text{Proper})$ and ω_1 inaccessible to reals. We shall show that either ω_1 is Mahlo in L (i.e., if the set of countable L-inaccessibles is stationary in L) or ω_1 is reflecting in L. This proves the Theorem: If ω_1 is Mahlo in L, then by the its inaccessibility in L, L_{ω_1} is a model of ZFC. For the same reason, the set $\{\alpha \mid \alpha < \kappa \text{ and } L_\alpha \prec L_\kappa\}$ is a closed unbounded subset of κ. Since ω_1 is Mahlo in L there is an L-inaccessible $\alpha < \kappa$ in this set, and for any such α, $L_\kappa \vDash \alpha$ is reflecting.

So assume that ω_1 is not Mahlo in L (and therefore that $0^\#$ does not exist). We will show that ω_1 is reflecting in L.

To show that ω_1 is reflecting in L it suffices to show: If x belongs to L_{ω_1}, φ is a formula and for some L-cardinal $\lambda \geq \omega_1$, $L_\lambda \vDash \varphi(x)$ then there is such a $\lambda < \omega_1$ with $x \in L_\lambda$. For, given this, if φ is a Σ_2 formula with parameter $x \in L_{\omega_1}$, then $L_\lambda \vDash \varphi(x)$ for some L-cardinal λ and therefore by assumption for some such $\lambda < \omega_1$; it follows that $L_{\omega_1} \vDash \varphi(x)$, as whenever $\lambda_0 < \lambda_1$ are L-cardinals, any Σ_2 formula with parameters from L_{λ_0} which is true in L_{λ_0} is also true in L_{λ_1}.

The proof proceeds in three steps:

Step 1. By a countably-closed forcing we produce $A \subseteq \omega_1$ such that every subset of ω_1 belongs to $L[A]$ and if $L_\alpha[A]$, $\alpha > \omega_1$ is a model of ZFC$-$Power, then $L_\alpha[A] \models$ There is an L-cardinal λ such that L_λ satisfies $\varphi(x)$.

Step 2. By a further proper forcing, we produce $A^* \subseteq \omega_1$ such that if $L_\alpha[A^* \cap \gamma]$ is any model of ZFC$-$Power satisfying $\gamma = \omega_1$, then $L_\alpha[A^* \cap \gamma] \models$ There is an L-cardinal λ such that L_λ satisfies $\varphi(x)$.

Step 3. By a further ccc forcing, we produce a real R such that for all α, if $L_\alpha[R]$ is a model of ZFC$-$Power in which "ω_1 exists", then $L_\alpha[R] \models$ There is an L-cardinal λ such that L_λ satisfies $\varphi(x)$.

This will complete the proof: The latter condition on the real R is unchanged if we restrict to countable α, by reflection. Therefore this condition is equivalent to a $\Pi_1(H(\omega_1))$ condition, and by $\mathrm{Abs}(\Sigma_2(H(\omega_1)), \mathrm{Proper})$ holds for some real R in V. By our assumption that ω_1 is inaccessible to reals, $L_{\omega_1}[R]$ satisfies "ω_1 exists" and therefore $L_{\omega_1}[R]$ satisfies that there is an L-cardinal λ such that $L_\lambda \models \varphi(x)$. Then λ really is an L-cardinal and therefore we have completed the proof that ω_1 is reflecting in L.

Now we turn to the proofs of Steps 1, 2 and 3.

Lectures 9 and 10

Now we turn to the proofs of Steps 1, 2 and 3.

Proof of Step 1. Choose an L-cardinal λ such that $L_\lambda \models \varphi(x)$. Let $\delta > \lambda$ be a singular strong limit cardinal of uncountable cofinality. Since $0^\#$ does not exist, we have $\delta^+ = (\delta^+ \text{ of } L)$ and $2^\delta = \delta^+$.

Now collapse δ to ω_1 using countable conditions: Conditions in $\mathrm{Coll}(\omega_1, \delta)$ are functions p from a countable ordinal into δ, ordered by extension. As there are only δ countable subsets of δ, this forcing has cardinality δ and therefore preserves cardinals greater than δ. It follows that δ^+ of L is the ω_2 of the extension. CH holds in $L[A_0]$ as we have collapsed $2^{\aleph_0} < \delta$ to ω_1 without adding reals. Also, each subset of ω_1 added by this forcing has a name of the form $\{(\check{\alpha}, p) \mid p \in X_\alpha,\ \alpha < \omega_1\}$, where each X_α is a subset of the forcing; as there are only $2^\delta = \delta^+$ such names, it follows that $2^{\omega_1} = \omega_2$ in the extension.

Let G be the generic function added by $\mathrm{Coll}(\omega_1, \delta)$ and define $A_0 \subseteq \omega_1$ by $\alpha \in A_0$ iff $g(\alpha_0) < g(\alpha_1)$, where $\alpha = \langle \alpha_0, \alpha_1 \rangle$ is a pairing function on ω_1. Then in every model of ZFC$-$Power of the form $L_\alpha[A_0]$, $\alpha > \omega_1$, there is a well-ordering of ω_1 of length δ and therefore we have $L_\alpha[A_0] \vDash$ There is an L-cardinal λ such that L_λ satisfies $\varphi(x)$. It remains to guarantee that every subset of ω_1 belong to $L[A_0]$.

In $V[A_0]$ we have $\omega_2 = (\delta^+)^L$ and $2^{\omega_1} = \omega_2$. In this model let $B \subseteq \omega_2$ code all subsets of ω_1. We code B by $A_1 \subseteq \omega_1$ via a countably closed almost disjoint forcing: In $L[A_0]$ choose $\langle b_\beta \mid \beta < \omega_2 \rangle$ to be distinct subsets of ω_1. We can assume that these sets are almost disjoint, in the sense that if $\beta_0 \neq \beta_1$, then b_{β_0} and b_{β_1} have countable intersection. Conditions in the coding of B by A_1 are pairs (p, p^*), where p is a countable subset of ω_1 and p^* is a countable subset of ω_2, ordered by:

$(p, p^*) \leq (q, q^*)$ iff p end-extends q, p^* contains q^* and $p - q$ is disjoint from b_α for $\alpha \in B \cap q^*$.

This forcing is countably closed, has the ω_2-cc and therefore preserves cardinals. Also if A_1 is the union of the first components of conditions in the generic, then we have:

$\alpha \in B$ iff A_1 is almost disjoint from b_α

and therefore B belongs to $L[A_1]$. As the generic for this forcing is entirely determined by the set A_1, it follows that every subset of ω_1 in $V[A_0][A_1]$ belongs to $L[A_0][A_1]$. So the desired set satisfying the requirement of Step 1 is $A = \{2\alpha \mid \alpha \in A_0\} \cup \{2\alpha + 1 \mid \alpha \in A_1\}$.

Proof of Step 2. We produce A^* using the following forcing. P consists of all $p : \gamma(p) \to 2$, $\gamma(p) < \omega_1$, such that:

$(*)$ For all $\gamma \leq \gamma(p)$ and all α, if $L_\alpha[A \cap \gamma, p \restriction \gamma]$ is a model of ZFC$-$Power where $\alpha > \gamma$ and γ is the ω_1 of $L_\alpha[A \cap \gamma, p \restriction \gamma]$ then $L_\alpha[A \cap \gamma, p \restriction \gamma] \vDash$ There is an L-cardinal λ such that $\varphi(x)$ holds in L_λ.

A P-generic adds a function $F : \omega_1 \to 2$ such that $A^* = \{2\beta \mid \beta \in A\} \cup \{2\beta + 1 \mid F(\beta) = 1\}$ satisfies Step 2, since this is guaranteed for countable γ by the definition of P and for $\gamma = \omega_1$ by Step 1. It remains to show:

Lemma 3.3.2. P is proper.

Proof of Lemma. It suffices to show that for CUB many countable $N \prec L_{\omega_2}[A]$, each condition p in N can be extended to a condition q such that q forces the generic to intersect $D \cap N$ whenever D is a dense set in N. We take all countable $N \prec L_{\omega_2}[A]$ which have A and x as elements. Suppose that p belongs to N and let N be isomorphic to $\bar{N} = L_\beta[A \cap \delta]$, where δ is the ω_1 of $\bar{N}$. Now N contains a witness C to the non-Mahloness of ω_1 in L, and since $C \cap \delta$ is unbounded in δ, it follows that δ belongs to C and is therefore singular in L. Therefore β is not an L-cardinal. Let μ be the least ordinal so that β is collapsed in L_μ.

We shall build q to be an extension of p of length δ, as the union of conditions of length less than δ. $(*)$ holds for q when γ of $(*)$ is less than δ due to the fact that q is the union of conditions of length less than δ. $(*)$ holds for q when γ of $(*)$ is equal to δ and α of $(*)$ is at most β, by the elementarity of N in $L_{\omega_2}[A]$. $(*)$ holds for q when γ of $(*)$ is equal to δ and α of $(*)$ is between β and μ, as in this case any L-cardinal of L_β is also an L-cardinal of L_α. Thus it suffices to build q so that δ is collapsed in $L_\mu[A \cap \delta, q]$, for then $(*)$ is vacuous when γ of $(*)$ is equal to δ and α of $(*)$ is at least μ.

As β is collapsed in $L_\mu[A \cap \delta]$ and we can assume that δ is not, we can write $L_\beta[A \cap \delta]$ as the union of a continuous chain $\langle M_i \mid i < \delta \rangle$ of Σ_1-elementary submodels of $L_\beta[A \cap \delta]$, where each M_i is countable in $L_\mu[A \cap \delta]$ and the chain itself belongs to $L_\mu[A \cap \delta]$. Let C be the set of intersections of the models of this chain with δ, a CUB subset of δ. We define an ω-sequence $p = p_0 \geq p_1 \geq \cdots$ of conditions below p such that each p_n belongs to N, each dense set in N is forced by some p_n to intersect the generic in N and if q is the union of the p_n's, then $\{\eta \in C \mid q(\eta) = 1\}$ is a cofinal subset of C of ordertype ω. Then δ is collapsed in $L_\mu[A \cap \delta, q]$, as desired.

To define the p_n's, enumerate the dense $D \in N$ in an ω-sequence $\langle D_n \mid n \in \omega \rangle$ and choose a cofinal subset C_0 of C of ordertype ω. Inductively, choose p_n as follows: If p_n is defined then first extend p_n at the next ω ordinals to code some $M_i \cap \delta \in C_0$, where both D_n and this extension belong to M_{i+1}. Then extend further to length $M_i \cap \delta$, always assigning the value 0. Finally, choose p_{n+1} to assign the value 1 at $M_i \cap \delta$ and belong to $D_n \cap M_{i+1}$. □ (Lemma 3.3.2)

Proof of Step 3. Now we code A by a real R. As ω_1 is not Mahlo in L, there is a CUB $C \subseteq \omega_1$, $C \in L$, consisting of L-singulars. Let $\langle \alpha_i \mid i < \omega_1 \rangle$ be the increasing enumeration of $C \cup \{0\}$ and for each i let R_i be a real coding the countable ordinal α_{i+1}. Then if we define B to be $\{\alpha_i + n \mid i < \omega_1$ and $n \in R_i\}$, we have: α countable $\to$ α countable in $L[B \cap \alpha]$. Using this, we choose distinct reals R_α, $\alpha < \omega_1$ so that R_α can be defined uniformly in $L[B \cap \alpha]$. We may assume that the R_α's are almost disjoint (mod finite). Now use these reals to code B, A^* by a real R using a ccc almost disjoint coding: A condition is a pair (p, p^*) where p is a finite subset of ω and p^* is a finite subset of ω_1, ordered by

$(p, p^*) \leq (q, q^*)$ iff p end-extends q, p^* contains q^* and $p - q$ is disjoint from $R_{2\alpha}$ when $\alpha \in B \cap q^*$ and disjoint from $R_{2\alpha+1}$ when $\alpha \in A^* \cap q^*$.

If R is the union of the first components of the generic, then R is almost disjoint from $R_{2\alpha}$ iff $\alpha \in B$ and is almost disjoint from $R_{2\alpha+1}$ iff $\alpha \in A^*$. The forcing is ccc and therefore preserves cardinals. Finally, as $A^* \cap \omega_1^{L_\alpha[R]}$ is definable in $L_\alpha[R]$ for each $\alpha < \omega_1$, R fulfills the condition of Step 3. □ (Theorem 3.3)

Lecture 11

We have seen that $\Sigma_2(H(\omega_1))$ absoluteness for proper forcings is consistent relative to ZFC, but in the presence of the additional assumption that ω_1 is inaccessible to reals, it has the consistency strength of a reflecting cardinal. If "proper" is weakened to "semiproper", the situation is the same, using a modification of the proof of Theorem 3.2.

A forcing P is *stationary-preserving* (at ω_1) iff stationary subsets of ω_1 remain stationary in P-generic extensions.

Theorem 3.4. Suppose that $\text{Abs}(\Sigma_2(H(\omega_1)), \text{stationary-preserving set-forcing})$ holds. Then ω_1 is inaccessible to reals.

Corollary 3.5. $\text{Abs}(\Sigma_2(H(\omega_1)), \text{stationary-preserving set-forcing})$ is equiconsistent with the existence of a reflecting cardinal.

Proof of Theorem 3.4. We first prove:

Lemma 3.4.1. If $0^\#$ does not exist then every set of ordinals is constructible from a real in a stationary-preserving set-forcing extension.

Proof. As in Step 1 of the proof of Theorem 3.3, we can produce $A \subseteq \omega_1$ by a countably-closed forcing so that in the extension $H(\omega_2) = L_{\omega_2}[A]$ and the given set of ordinals belongs to $H(\omega_2)$. Let P be the "reshaping forcing", whose conditions are $p : |p| \to 2$, $|p| < \omega_1$ such that for all $\alpha \leq |p|$, α is countable in $L[A \cap \alpha, p \restriction \alpha]$. We will show that P is stationary-preserving. Assuming this, let G be P-generic and $F : \omega_1 \to 2$ the union of the conditions in G. Using F, we can choose a sequence $\langle R_\alpha \mid \alpha < \omega_1 \rangle$ of distinct reals such that R_α is definable uniformly in $L[A \cap \alpha, F \restriction \alpha]$ (by taking R_α to be the least real in $L[A \cap \alpha, F \restriction \alpha]$ distinct from the R_β, $\beta < \alpha$). Now as in Step 3 of the proof of Theorem 3.3, we can code A, G by a real via a ccc forcing, resulting in a stationary-preserving extension in which the given set of ordinals is constructible from a real, as desired.

Now we show that P is stationary-preserving. Given $p \in P$, a stationary $X \subseteq \omega_1$ and a name σ for a CUB subset of ω_1, let C be a CUB subset of ω_1 such that:

1. If α is in C and β is less than α then p is in $L_\alpha[A]$ and every $q \leq p$ in $L_\alpha[A]$ has an extension $r \in L_\alpha[A]$ such that $r \Vdash \beta^* \in \sigma$ for some β^* between β and α.
2. If α is in C then $C \cap \alpha$ belongs to $L[A \cap \alpha]$.

C is constructed by choosing $L_\gamma[A]$, $\gamma > \omega_1$, to contain p, σ and A and taking C to be $\{i < \omega_1 \mid i = \omega_1 \cap M_i$, where $M_i =$ the Skolem hull of $i \cup \{p, \sigma, A\}$ in $L_\gamma[A]\}$.

Now choose $\alpha \in \text{Lim } C \cap X$ and let $\langle \gamma_n \mid n \in \omega \rangle$ be any increasing ω-sequence contained in C with supremum α. We inductively define conditions q_n of length γ_n as follows. Set q_0 to be the $L[A]$-least extension of p of length γ_0. If q_n is defined, let q'_n be the $L[A]$-least extension of q_n such that $q'_n(\gamma_n) = 1$ and q'_n forces some β_n greater than γ_n to belong to σ; note that by property 1 above, $\gamma'_n =$ (the length of q'_n) is less than the least element of C greater than γ_n. Let R_n be a real coding the ordinal γ_{n+1} and extend q'_n to q''_n of length $\gamma'_n + \omega$ by defining $q''_n(\gamma'_n + k) = R_n(k)$. Then q_{n+1} is obtained by extending q''_n to length γ_{n+1}, always taking the value 0 at and above $\gamma'_n + \omega$. It is clear that q_{n+1} is a condition, using the definition of q''_n.

Let q be the union of the q_n's. Then $\{\gamma \in C \cap [\gamma_0, \alpha) \mid q(\gamma) = 1\}$ equals $\{\gamma_n \mid n \in \omega\}$. By property 2 above, $\{\gamma_n \mid n \in \omega\}$ belongs to $L[A \cap \alpha, q]$, and therefore α is countable in $L[A, q]$, establishing that q is a condition. As q

forces that $\sigma \cap \alpha$ is unbounded in α, q also forces that α belongs to σ. Since α belongs to X, we have $q \Vdash X \cap \sigma \neq \emptyset$, as desired. □ (Lemma 3.4.1)

Note that Lemma 3.4.1 also holds under the weaker hypothesis that $R^\#$ does not exist for some real R, by relativisation to R. (Indeed, one only needs that $A^\#$ does not exist for some set of ordinals A.)

Lectures 12 and 13

Now to prove Theorem 3.4, suppose that ω_1 is not inaccessible to reals. Thus for some real R, $\omega_1 = \omega_1$ of $L[R]$. As the real R plays no role in the proof below, we will assume that R equals 0. In particular $0^\#$ does not exist and therefore by Lemma 3.4.1, in a stationary-preserving set-generic extension, $H(\omega_2) = L_{\omega_2}[R]$ for some real R. For the moment, argue in this extension. As the real R plays no role in the arguments below, we also assume that R equals 0.

For any $A \subseteq \omega_1$ consider now the function $f_A : \omega_1 \to \omega_1$ defined by

$f_A(\alpha)$ = the least β such that α is countable in $L_{\beta+1}[A \cap \alpha]$.

Note that by assumption, $\omega_1 = \omega_1^L$ and therefore f_A is totally defined for every A. We say that A is *faster than* B iff $f_A < f_B$ on a CUB.

Lemma 3.4.2. (Ralf Schindler) For any A there is a faster B in a further stationary-preserving forcing extension.

Given this lemma, we prove Theorem 3.4. Set $A_0 = R_0 = \emptyset$. By the lemma there is A_1 which is faster than A_0 in a stationary-preserving forcing extension. A_1, together with a CUB set C_1 witnessing that A_1 is faster than A_0, can be coded by a real R_1 via a ccc forcing; we write $A_1 = A(R_1)$, $C_1 = C(R_1)$. Then R_1 satisfies the $\Pi_1(H(\omega_1))$ condition

For all $\alpha < \omega_1$, $f_{A(R_1)}(\alpha) < f_\emptyset(\alpha)$ for all α in the CUB set $C(R_1)$.

By $\Sigma_2(H(\omega_1))$ absoluteness for stationary-preserving forcings, there is such a real R_1 in the original ground model V. Then the real R_1 is faster than the real $R_0 = \emptyset$. But we can repeat this, obtaining R_{n+1} which is faster than R_n, for each n. Thus $f_{R_{n+1}} < f_{R_n}$ on a CUB for each n, a contradiction.

Proof of Lemma 3.4.2. The proof is similar to the proof that the reshaping forcing is stationary-preserving. Consider the forcing P whose conditions are pairs (b, c) where:

c is a countable closed subset of ω_1.
$b : \max c \to 2$.
For all $\alpha \in c$, α is countable in $L_{f_A(\alpha)}[b \restriction \alpha]$.

Conditions are ordered by: $(b_0, c_0) \leq (b_1, c_1)$ iff c_0 end-extends c_1 and $b_0 \cap \max c_1 = b_1$. Any condition can be extended so as to increase $\max c$ above any given countable ordinal: Given (b, c) there are arbitrary large limit ordinals $\alpha > \max c$ with $f_A(\alpha) > \alpha$. We obtain a condition by adding α to c and extending b to any b' of length α so that α is countable in $L_{\alpha+1}[b']$.

Thus if G is P-generic then $B = \cup\{b \mid (b, c) \in G \text{ for some } c\}$ is faster than A, as witnessed by the CUB set $C = \cup\{c \mid (b, c) \in G \text{ for some } b\}$. It remains only to show that P is stationary-preserving.

Suppose that $p = (b, c) \in P$, X is stationary and σ is a name for a CUB. Let $C_0 \supseteq C_1$ be CUB sets such that:

1. If α is in C_0 and β is less than α then p is in L_α and every $q \leq p$ in L_α has an extension $r \in L_\alpha$ such that $r \Vdash \beta^* \in \sigma$ for some β^* between β and α.
2. If α is in C_1 then $f_A(\alpha) > \alpha$ and $C_0 \cap \alpha$ belongs to $L_{f_A(\alpha)}$.

C_0 is constructed by choosing L_γ, $\gamma > \omega_1$, to contain p, σ, A and taking C_0 to be $\{i < \omega_1 \mid i = \omega_1 \cap M_i$, where $M_i =$ the Skolem hull of $i \cup \{p, \sigma, A\}$ in $L_\gamma\}$. Then C_1 is defined to be $\{i < \omega_1 \mid i = \omega_1 \cap N_i$, where $N_i =$ the Skolem hull of $i \cup \{p, \sigma, A, \gamma\}$ in $L_{\gamma+\omega}\}$.

Now choose $\alpha \in \text{Lim } C_1 \cap X$ and let $\langle \gamma_n \mid n \in \omega \rangle$ be any increasing ω-sequence contained in C_1 with supremum α. We inductively define conditions $q_n = (b_n, c_n)$ of length γ_n as follows. Set q_0 to be the L-least extension of p of length γ_0. If q_n is defined, let $q'_n = (b'_n, c'_n)$ be the L-least extension of q_n such that $b'_n(\gamma_n) = 1$ and q'_n forces some β_n greater than γ_n to belong to σ; note that by property 1 above, $\gamma'_n =$ (the length of q'_n) is less than the least element of C_0 greater than γ_n. Let R_n be a real coding the ordinal γ_{n+1} and extend b'_n to b''_n of length $\gamma'_n + \omega$ by defining $b''_n(\gamma'_n + k) = R_n(k)$ for each $k \in \omega$. Then q_{n+1} is obtained by setting $c_{n+1} = c'_n \cup \{\gamma_{n+1}\}$ and extending

b''_n to length γ_{n+1}, always taking the value 0 at and above $\gamma'_n + \omega$. Note that q_{n+1} is a condition as γ_{n+1} is countable in $L_{\gamma_{n+1}+1}[b_{n+1}]$ but $f_A(\gamma_{n+1}) > \gamma_{n+1}$.

Let b be the union of the b_n's and c the union of the c_n's together with the ordinal α. Then $\{\gamma \in C_0 \cap [\gamma_0, \alpha) \mid b(\gamma) = 1\}$ equals $\{\gamma_n \mid n \in \omega\}$ and by property 2 above, $C_0 \cap \alpha$ belongs to $L_{f_A(\alpha)}$. It follows that α is countable in $L_{f_A(\alpha)}[b]$, establishing that $q = (b, c)$ is a condition. As q forces that $\sigma \cap \alpha$ is unbounded in α, q also forces that α belongs to σ. Since α belongs to X, we have $q \Vdash X \cap \sigma \neq \emptyset$, as desired. $\square$

This completes the proof of Theorem 3.

Persistence of $\Sigma_3(H(\omega_1))$ absoluteness

It is reasonable to consider $\text{Abs}(\Sigma_3(H(\omega_1)), \mathcal{P})$ provided one imposes the hypothesis that $\Sigma_3(H(\omega_1))$ formulas persist for $\mathcal{P}$-generic extensions. The latter is equivalent to saying that $\text{Abs}(\Sigma_2(H(\omega_1)), \mathcal{P})$ holds in all $\mathcal{P}$-generic extensions, a form of "two-step absoluteness" for $\mathcal{P}$-forcing. We consider next some examples of this.

Theorem 6.1. The following are equivalent:
1. All set-generic extensions obey $\Sigma_2(H(\omega_1))$-absoluteness for further set-generic extensions.
2. Every set of ordinals has a #.

Proof. ($1 \to 2$) Assume property 1 and we first show that $0^\#$ exists. If not, then $\kappa^+ = \kappa^*$ of L, where $\kappa = \aleph_\omega$. Let $V[G]$ be a set-generic extension where κ^+ of $L = \omega_1$, obtained by collasping κ to ω. Then for some real R in $V[G]$, $\omega_1 = \omega_1$ of $L[R]$. This is a $\Pi_2(H(\omega_1))$ property:

$\omega_1 = \omega_1$ of $L[R]$ iff
$H(\omega_1) \vDash \forall\alpha\exists S(S$ is a real in $L[R]$ and S codes $\alpha)$.

But this property is false in $V[G][H]$, where H collapses ω_1 of $L[R]$ to ω. So $\Sigma_2(H(\omega_1))$ absoluteness fails between $V[G]$ and $V[G][H]$.

The same argument shows that $R^\#$ exists for each real R. As property 1 holds in all set-generic extensions, it follows that in all set-generic extensions, every real has a #, i.e., every set of ordinals has a #.

Lectures 14 and 15

(2 $\to$ 1) Recall that elements of $H(\omega_1)$ can be coded by reals, and the set of reals $\mathcal{C}$ coding an element of $H(\omega_1)$ forms a Π^1_1 set (i.e., a set of the form $\{x \mid \forall y \varphi(x, y)\}$ where x, y vary over reals and φ is arithmetical). It follows that a $\Sigma_2(H(\omega_1))$ formula can be translated into a Σ^1_3 formula about reals:

$\exists a \in H(\omega_1) \forall b \in H(\omega_1) \varphi(a, b)$ (φ Δ_0) iff
$\exists x \in \mathcal{C} \forall y \in \mathcal{C} \varphi^*(x, y)$,

where φ^* is arithmetical. As $\mathcal{C}$ is Π^1_1 the latter formula is Σ^1_3. So property 1 of the theorem follows from:

($*$) All set-generic extensions obey Σ^1_3 absoluteness with respect to further set-generic extensions.

We will prove ($*$) under the assumption that every set has a #, or equivalently, that in every set-generic extension, every real has a #.

First just assume that every real has a # and let $A = \{x \mid \forall y \exists z \varphi(x, y, z)\}$, φ arithmetical, be a Π^1_2 set. Assuming that A is nonempty, we show how to choose a "canonical" element of A.

A *tree* on a set B is a collection of finite sequences of elements of B closed under initial segment. If T is a tree on $B_1 \times B_2 \times \cdots \times B_n$, s_i a finite sequence from B_i for $1 \leq i < n$ and the s_i's all have the same length, then $T(s_1, \ldots, s_{n-1}) = \{t_n \mid (s_1 \restriction l, \ldots, s_{n-1} \restriction l, t_n) \in T$, where $l =$ length of $t_n \leq$ length of each $s_i\}$ (and where we identify an n-tuple of sequences of length l with a sequence of length l of n-tuples in the natural way). If x_i is an ω-sequence from B_i for each $1 \leq i < n$ then $T(x_1, \ldots, x_{n-1}) = \cup\{T(x_1 \restriction l, \ldots, x_{n-1} \restriction l) \mid l < \omega\}$.

Now $B = \{(x, y) \mid \exists z \varphi(x, y, z)\}$ is Σ^1_1 and therefore there is a tree T on $2 \times 2 \times \omega$ such that $(x, y) \in B$ iff $T(x, y)$ has an infinite branch. Then:

$x \in A$ iff
$\forall y$ $T(x, y)$ has an infinite branch.

Now let κ be an uncountable regular cardinal and define the orderings U^κ and $U^\kappa(x)$ (x a real) as follows: An element of U^κ is a triple (s, t, f), with s and t finite sequences of 0's and 1's of the same length and f an order-preserving function from $(T(s, t)^*, <^*)$ into κ, where $T(s, t)^*$ is the finite set of all finite sequences in $T(s,t)$ taking values less than Length(s) = Length(t), and where $<^*$ is the Kleene-Brouwer order on finite sequences of natural numbers: $u <^* v$ iff u properly extends v or u is less than v in the lexicographic order. The ordering on U^κ is the natural one: $(s_0, t_0, f_0) \leq (s_1, t_1, f_1)$ iff s_0, t_0, f_0 extend s_1, t_1, f_1, respectively. For a real x, $U^\kappa(x)$ denotes the set of pairs (t, f) such that for some n, $(x \restriction n, t, f)$ belongs to U^κ.

Claim 1. $x \in A$ iff $U^\kappa(x)$ is well-founded.

Proof of Claim 1. An infinite descending sequence through $U^\kappa(x)$ yields a real y and an order-preserving function from $(T(x,y), <^*)$ into κ; it follows that $T(x,y)$ has no infinite branch, and therefore x does not belong to A. Conversely, if x does not belong to A, then choose y such that $T(x,y)$ has no infinite branch, choose an order-preserving function f from the countable well-ordering $(T(x,y), <^*)$ into κ and define $f_n = f \restriction T(x \restriction n, y \restriction n)^*$; then $(x \restriction n+1, y \restriction n+1, f_{n+1})$ is less than $(x \restriction n, y \restriction n, f_n)$ in U^κ for each n, so $(y \restriction n+1, f_{n+1})$ is less than $(y \restriction n, f_n)$ in $U^\kappa(x)$ for each n; it follows that $U^\kappa(x)$ is not well-founded. □

Note that if x belongs to A then the canonical ranking function F^x on $U^\kappa(x)$ is constructible from a real, as it is constructible from x and T. Now we want to choose a particular x such that $U^\kappa(x)$ is well-founded. For this purpose we need to compare ranking functions on the orderings $U^\kappa(s,t) = \{f \mid (s \restriction n, t \restriction n, f) \in U^\kappa$ for some $n\}$. Fix s, t of the same length and let L^* denote $\cup\{L[x] \mid x$ a real$\}$. Suppose that $F, G \in L^*$ are functions from $U^\kappa(s,t)$ into the ordinals. We write $F \leq^* G$ iff from some CUB $C \subseteq \kappa$, $C \in L^*$, $F(f) \leq G(f)$ for all $f \in U^\kappa(s,t)$ with Range $(f) \subseteq C$. For any F, G either $F \leq^* G$ or $G \leq^* F$, since F, G are constructible from reals and therefore by our assumption that every real has a #, there is a CUB subset of κ which forms a set of order-indiscernibles relative to F, G. Therefore $\leq^*$ gives a wellordering if we identify F with G when $F =^* G$.

Given n let $t_1, t_2, \ldots, t_{2^n}$ list the $0,1$-sequences of length n in lexicographic order. Then define $\alpha^x_n = \langle \beta_1, \ldots, \beta_{2^n} \rangle$, where β_i is the rank of $F^x \restriction U^\kappa(x \restriction n, t_i)$ in $\leq^*$.

We now define a canonical element of A. Choose x_1 to minimize $\alpha^x_1, x(0)$ (in the lexicographic ordering of finite sequences of ordinals) for $x \in A$ and set $n_0 = x_1(0)$. Then choose x_2 to minimize $\alpha^x_2, x(1)$ for $x \in A$ which minimize

$\alpha_1^x, x(0)$ and set $n_1 = x_2(1)$. Continue in this way, producing a real $x^* = \langle n_0, n_1, \ldots \rangle$.

Claim 2. $x^* \in A$.

Proof of Claim 2. For each n and t of length n choose $F^n(t)$ with domain $U^\kappa(x^* \restriction n, t)$ so that the ranks of the $F^n(t)$ realise $\alpha_n^{x_n}$. Then for some CUB C, the $F^n(t)$ restricted to elements of $U^\kappa(x^* \restriction n, t)$ with range in C cohere with each other. It follows that $U(x^*)$ is well-founded, and therefore x^* belongs to A. □

Now we are ready to verify $(*)$ (and therefore property 1 of the theorem), assuming that in every set-generic extension, every real has a #. Suppose that $V[G]$ is a set-generic extension of V and $\varphi(x)$ is a Π_2^1 formula with real parameter from $V[G]$. Suppose that $V[G][H]$ is a set-generic extension of $V[G]$ where $\varphi(x)$ holds for some real x. We want to show that $\varphi(x)$ holds in $V[G]$ for some x in $V[G]$. By assumption every real in $V[G][H]$ has a #. Let κ be greater than the size of the forcing that produces H over $V[G]$. Now form the ordering U^κ as above in $V[G]$, for the Π_2^1 set $A = \{x \mid \varphi(x)\}$. U^κ has the same definition in $V[G][H]$ as it has in $V[G]$. Any ranking function on $U^\kappa(s,t)$, s, t finite $0,1$-sequences of the same length, which is constructible from a real in $V[G][H]$ is $=^*$-equivalent to such a function in $V[G]$ (with a CUB $C \subseteq \kappa$ in $V[G]$ witnessing this), as H is generic over $V[G]$ for a forcing of size less than κ.

Now in $V[G][H]$, consider the set of pairs (s, F), where s is a finite $0,1$-sequence and F is a ranking function on $U^\kappa(s)$ constructible from a real in $V[G]$. Order such pairs by $(s_0, F_0) \leq (s_1, F_1)$ iff s_0 extends s_1 and F_0 extends F_1 on all (t, f) with Range $(f) \subseteq C$, for some CUB $C \subseteq \kappa$ in $V[G]$. Then in $V[G][H]$ this ordering is not well-founded, as $U^\kappa(x)$ has a ranking function constructible from a real for some x, and the restriction of this function to $U^\kappa(x \restriction n)$ is $=^*$-equivalent to a function constructible from a real in $V[G]$, witnessed by a CUB $C \subseteq \kappa$ in $V[G]$. It follows that this ordering is not well-founded in $V[G]$, $U^\kappa(x)$ is well-founded for some real x in $V[G]$ and the given Π_2^1 formula $\varphi(x)$ holds for some real in $V[G]$, as desired. □

Lectures 16 and 17

Theorem 9. The following are equiconsistent:
1. $\Sigma_3(H(\omega_1))$-absoluteness for set-generic extensions and every set has a #.
2. There exists a reflecting cardinal and every set has a #.

Proof. We imitate the proof of Theorem 3.1. Suppose that every set has a # and κ is reflecting. Let $V[G]$ be the generic extension of V obtained by collapsing every ordinal less than κ to ω; we show that $V[G]$ witnesses property 1. Suppose that φ is a $\Sigma_3(H(\omega_1))$ formula with parameter from $V[G]$ which is forced to hold in some set-generic extension of $V[G]$. First assume that the parameter in φ belongs to V. Then the following Σ_2 statement mentioning this parameter holds in V:

There is a cardinal δ and a forcing $P \in H(\delta)$ such that $H(\delta) \vDash (P \Vdash \varphi)$.

By reflection there is such a δ, P in $H(\kappa)$. Let $V[g]$ be P-generic over V, $g \in V[G]$; there is such a g since the V-power set of P is countable in $V[G]$. Then $V[g]$ satisfies φ. Since $V \vDash$ Every set has a #, φ is persistent for set-generic extensions of V and therefore φ also holds in $V[G]$. Since $V[G]$ also satisfies "Every set has a #", we are done. If the parameter in φ does not belong to V, then as in the proof of Theorem 3.1, we factor $V[G]$ as $V[G(<\alpha)][G(\geq \alpha)]$, where the parameter belongs to $V[G(<\alpha)]$, $\alpha < \kappa$.

Now assume that 1 holds. We show that ω_1 is reflecting in an appropriate inner model where every set has a #.

Fact. Suppose that every set has a #. Then there is a smallest inner model $L^\#$ in which every set has a #. Moreover, this inner model has the following property: There is a sequence $\langle L^\#_\alpha \mid \alpha \in \mathrm{Ord}\rangle$, such that:

1. For each α, $L^\#_\alpha$ is transitive of ordinal height α.
2. $\alpha \leq \beta \rightarrow L^\#_\alpha \subseteq L^\#_\beta$.
3. For each infinite $L^\#$-cardinal θ, $L^\#_\theta = H(\theta)^{L^\#}$.
4. For each infinite cardinal θ, $\langle L^\#_\alpha \mid \alpha < \theta\rangle$ is Σ_2-definable over $H(\theta)$.

Assuming 1, we now show that $\kappa = \omega_1^V$ is reflecting in $L^\#$. Suppose that $L^\# \vDash \varphi$, where φ is a Σ_2 formula with parameters from $L^\#_\kappa$. We must show that φ is true in $L^\#_\kappa$. Since φ is true in $L^\#$, by reflection it is also true in $L^\#_\theta$ for some $L^\#$-cardinal θ. There is a set-generic extension of V in which

θ is countable. Therefore in some set-generic extension of V the following formula (with parameters from $L^\#_\kappa \subseteq H(\omega_1)^V$) is true:

There is a countable ordinal θ such that θ is a cardinal of $L^\#$ and $L^\#_\theta \vDash \varphi$.

This formula is $\Sigma_3(H(\omega_1))$ as $\langle L^\#_\alpha \mid \alpha < \omega_1\rangle$ is Σ_2-definable over $H(\omega_1)$. By our assumption of $\Sigma_3(H(\omega_1))$-absoluteness, the above formula is also true in V. Therefore there is an ordinal θ less than $\omega_1^V = \kappa$ such that $L^\#_\theta \vDash \varphi$ and $H(\omega_1)^V \vDash \theta$ is an $L^\#$-cardinal. Then θ really is an $L^\#$-cardinal and therefore $L^\#_\theta$ is Σ_1-elementary in $L^\#_\kappa$. As φ is Σ_2, it follows that $L^\#_\kappa$ also satisfies φ, as desired. □

$\Sigma_4(H(\omega_1))$-absoluteness for set-generic extensions is reasonable provided $\Sigma_4(H(\omega_1))$ formulas persist for set-generic extensions, i.e., provided that all set-generic extensions obey $\Sigma_3(H(\omega_1))$-absoluteness for further set-generic extensions.

Theorem 10. Assume that n is greater than 0. Then the following are equiconsistent:
1. All set-generic extensions obey $\Sigma_{n+2}(H(\omega_1))$-absoluteness for further set-generic extensions.
2. There exist n strong cardinals.

Proof. We first show that the consistency of 2 implies that of 1.

Definition. Suppose that $\kappa < \lambda$ are inaccessibles. Then κ is λ-*strong* iff there is an elementary embedding $j : V \to M$ with critical point κ such that $H(\lambda) \subseteq M$. And κ is *strong* iff it is λ-strong for all inaccessible $\lambda > \kappa$.

Fact. If κ is λ-strong then there is an elementary $j : V \to M$ witnessing this such that $M^\kappa \subseteq M$.

We shall need some facts about trees. A *tree* on a set X is a subset of $\mathrm{Seq}(X)$ = the set of finite sequences of elements of X closed under initial segments. For T a tree on X we let $[T]$ denote the set of infinite branches through T, i.e., the set of $f \in X^\omega$ such that $f \restriction n \in T$ for all n. We think of a tree on $Y \times Z$ as a set of pairs $(s,t) \in \mathrm{Seq}(Y) \times \mathrm{Seq}(Z)$ where s and t have the same length. If T is a tree on $Y \times Z$ and $s \in \mathrm{Seq}(Y)$ then we set $T_s = \{t \mid (s \restriction \mathrm{Length}(t), t) \in Y\}$ and for $x \in Y^\omega$, $T_x = \cup\{T_{x\restriction n} \mid n \in \omega\}$. The projection $p[T]$ is defined by: $x \in p[T]$ iff T_x has an infinite branch. We say that $p[T]$ is *Z-Suslin* via T.

Now we consider *κ-absolute* Suslin representations. We say that a set G is *$(< \kappa)$-generic* over a model M iff G is P-generic over M where $M \models P$ has cardinality less than κ. Suppose that T, U are trees on $X \times Y$, $X \times Z$, respectively. We say that T, U are *κ-absolute complements* iff whenever G is $(< \kappa)$-generic over V, we have $V[G] \models p[T] = X^\omega - p[U]$. The tree T is *κ-absolutely complemented* iff there is a U such that T, U are κ-absolute complements.

Remark. Note that if $p[T]$, $p[U]$ are disjoint in V then they are automatically disjoint in any extension of V, by a simple absoluteness argument. What absolute complementation adds is that the union of $p[T]$, $p[U]$ is all of X^ω.

Definition. $A \subseteq X^\omega$ is *κ-absolutely Suslin* iff $A = p[T]$ for some κ-absolutely complemented tree T. If A is defined by the formula φ (with parameters), then the pair (A, φ) is *κ-absolutely Suslin* iff $A = p[T]$ for some κ-absolutely complemented tree T with the additional property that $p[T] = \{x \mid \varphi(x)\}$ in all $(< \kappa)$-generic extensions. We say that A is *absolutely Suslin* iff A is κ-absolutely Suslin for every κ; similarly for (A, φ).

The proof of Theorem 6.1 $(2 \to 1)$ shows:

Theorem 10.1. If every set has a # then (A, φ) is absolutely Suslin whenever $A \subseteq \omega^\omega$ is the set of reals defined by a Σ^1_3 formula φ (with real parameters).

We shall prove:

Theorem 10.2. Suppose that there exists a strong cardinal κ, $A \subseteq \omega^\omega \times \omega^\omega$ and (A, φ) is absolutely Suslin. Let $B = \{x \mid (x, y) \in A \text{ for some } y\}$ and ψ the formula $\exists y \in \omega^\omega \varphi(x, y)$. Then (B, ψ) is absolutely Suslin in $V[G]$, where G is generic for the Lévy collapse of 2^{2^κ} to ω.

Now using this, we prove that the consistency of 2 implies that of 1. Suppose that κ is the least strong cardinal of V. It follows that V is closed under #, and therefore by Theorem 10.1, (A, φ) is absolutely Suslin when A is the set of reals defined by a Σ^1_3 formula φ. The same is true for Π^1_3. By Theorem 10.2, after collapsing 2^{2^κ} to ω, we obtain a model where (B, ψ)

is absolutely Suslin when B is defined by a Σ^1_4 formula ψ. In particular, for each κ there is a tree T_κ such that $p[T_\kappa] = \{x \mid \psi(x)\}$ in all $(< \kappa)$-generic extensions; this implies Σ^1_4-absoluteness in all set-generic extensions, by the absoluteness of well-foundedness for trees. If there were two strong cardinals in V, then there is still a strong cardinal in this generic extension, and we can repeat the argument, obtaining a model where Σ^1_5-absoluteness holds for set-generic extensions, etc. As Σ^1_{n+3}-absoluteness is the same as $\Sigma_{n+2}(H(\omega_1))$-absoluteness, we are done.

Lecture 18

Proof of Theorem 10.2. Suppose that A is the projection of the tree T on $\omega \times \omega \times Z$, where T has a λ-absolute complement U and $p[T] = \{x \mid \varphi(x)\}$ in all $(< \lambda)$-generic extensions. Let S be the same as the tree T, but regarded as a tree on $\omega \times (\omega \times Z)$. So $p[S] = \{x \mid (x, y) \in p[T]$ for some $y\} = B$, and $p[S]$ equals $\{x \mid \exists y \varphi(x, y)\}$ in all $(< \lambda)$-generic extensions.

Claim. Suppose that κ is λ-strong and $j : V \to M$ witnesses this, where $M^\omega \subseteq M$. Suppose that T is a tree on $\omega \times Z$ for some Z. Let G be generic over V for the Lévy collapse of 2^{2^κ} to ω. Then in $V[G]$, $j(T)$ has a λ-absolute complement.

We prove Theorem 10.2 using this Claim. Applying the Claim to the tree S, we obtain a λ-absolute complement for $j(S)$ in $V[G]$, where G is generic for the Lévy collapse of 2^{2^κ} to ω. It suffices to show that $p[S] = p[j(S)]$ in $V[G][H]$ for any $(< \lambda)$-generic H, for then S has the same λ-absolute complement as $j(S)$. Argue now in $V[G][H]$. As $p[S] = \{x \mid (x, y) \in p[T]$ for some $y\}$, $p[j(S)] = \{x \mid (x, y) \in p[j(T)]$ for some $y\}$, it suffices to show that $p[T] = p[j(T)]$. Clearly $p[T] \subseteq p[j(T)]$, as j sends a branch through T to a branch through $j(T)$. Conversely, if $(x, y) \notin p[T]$, then $(x, y) \in p[U]$ (where U is a λ-absolute complement for T), so $(x, y) \in p[j(U)]$; by elementarity $p[j(U)]$ and $p[j(T)]$ are disjoint in M, and therefore by absoluteness are really disjoint. Therefore $(x, y) \notin p[j(T)]$.

To prove the Claim we shall need some facts about measures. For any set Z, $\mathrm{Meas}_\kappa(Z)$ denotes the set of κ-additive measures on $Z^{<\omega}$. If κ is ω_1 then we write $\mathrm{Meas}(Z)$ for $\mathrm{Meas}_\kappa(Z)$. If μ belongs to $\mathrm{Meas}(Z)$ then the *dimension* of μ, written $\dim(\mu)$, is the unique n such that $\mu(Z^n) = 1$. If $\mu, \nu \in \mathrm{Meas}(Z)$ then we say that μ *projects to* ν iff $\dim(\nu) \leq \dim(\mu)$ and for $A \subseteq Z^\omega$:

$\nu(A) = \mu(\{u \in Z^\omega \mid u \restriction \dim(\nu) \in A\})$. If μ projects to ν then there is a natural embedding $\pi_{\nu,\mu} : \mathrm{Ult}(V, \nu) \to \mathrm{Ult}(V, \mu)$ obtained by sending $[f]_\nu$ to $[f^*]_\mu$, where $f^*(u) = f(u \restriction \dim(\nu))$ for all $u \in Z^\omega$.

A *tower of measures* on Z is a sequence $\langle \mu_n \mid n < \omega \rangle$ such that $\mu_n \in \mathrm{Meas}(Z)$ has dimension n for each n, and whenever $m \leq n < \omega$, μ_n projects to μ_m. If $\langle \mu_n \mid n < \omega \rangle$ is a tower of measures then $\mathrm{Ult}(V, \langle \mu_n \mid n < \omega \rangle)$ denotes the direct limit of the $\mathrm{Ult}(V, \mu_n)$ via the embeddings π_{μ_m,μ_n}. One can show that $\mathrm{Ult}(V, \langle \mu_n \mid n < \omega \rangle)$ is well-founded iff whenever $\mu_{x \restriction n}(A_n) = 1$ for each n there exists f such that $f \restriction n \in A_n$ for each n.

Proof of Claim. There is a tree $T^* \subseteq T$ of size κ such that $p[T] = p[T^*]$ in any $(< \kappa)$-generic extension of V (obtained by listing all P, σ where $P \in H(\kappa)$ and σ is a P-name for a real, and for each such P, σ putting into T^* all elements of Z which are forced by some condition in P to belong to the least branch through T projecting to σ). We can assume that T^* is a tree on $\omega \times \kappa$.

In $V[G]$ the set $\mathrm{Meas}_\kappa(\kappa^{<\omega})$ is countable. Let $m : \omega \to j[\mathrm{Meas}_\kappa(\kappa^{<\omega})]$ be an enumeration in $V[G]$ such that $m(e)$ concentrates on κ^n for some $n \leq e$. Each measure in $j[\mathrm{Meas}_\kappa(\kappa^{<\omega})]$ extends from M to $M[G]$ since 2^{2^κ} is less than $j(\kappa)$. Similarly, since $\lambda \leq j(\kappa)$, these measures extend to $M[G][H]$ whenever H is $(< \lambda)$-generic over $M[G]$. Notice that since M contains $H(\lambda)$ and $M^\omega \subseteq M$, any $(< \lambda)$-generic H over $V[G]$ is in fact $(< \lambda)$-generic over $M[G]$ and $M[G][H]$ is ω-closed in $V[G][H]$.

Define the tree S to consist of all $(s, \langle \alpha_0 \ldots, \alpha_{n-1} \rangle)$ such that:

$s \in \omega^n$
$\alpha_0 < j(\kappa)^+$
For all $i < e < n$: If $m(e)$ concentrates on $j(T^*)_s$ and $m(e)$ projects to $m(i)$, then $\alpha_e < \pi_{m(i),m(e)}(\alpha_i)$.

We will show that S is a λ-absolute complement for $j(T)$ in $V[G]$. Let H be $(< \lambda)$-generic over $V[G]$ and x a real in $V[G][H]$; we must show that in $V[G][H]$, $x \in p[j(T)]$ iff $x \notin p[S]$. Note that since $M[G][H]$ is ω-closed in $V[G][H]$, x belongs to $M[G][H]$.

For each $(s, t) \in j(T^*)$, consider the measure $\Sigma(s, t)$ concentrating on T^*_s given by: $A \in \Sigma(s, t)$ iff $t \in j(A)$. Suppose that (x, f) is a branch through

$j(T^*)$ in V. Then $\mathrm{Ult}(V, \langle \Sigma(x \restriction n, f \restriction n) \mid n \in \omega \rangle)$ is well-founded: Otherwise, we can choose $A_n \in \Sigma(x \restriction n, f \restriction n)$ and $g_n : A_n \to \mathrm{Ord}$ such that for each n, $g_{n+1}(y) < g_n(y \restriction n)$ for $y \in A_{n+1}$. But then $j(g_{n+1})(f \restriction n+1) < j(g_n)(f \restriction n)$ for each n, contradiction. It follows that $\mathrm{Ult}(M, \langle j(\Sigma(x \restriction n, f \restriction n)) \mid n \in \omega \rangle)$ is well-founded. The measures $j(\Sigma(x \restriction n, f \restriction n))$ lift from M to $M[G][H]$ and therefore we have the well-foundedness of $\mathrm{Ult}(M, \langle j(\Sigma(x \restriction n, f \restriction n)) \mid n \in \omega \rangle)$ for any branch (x, f) through $j(T^*)$ in $M[G][H]$; note that any branch through $j(T^*)$ in $V[G][H]$ in fact belongs to $M[G][H]$ as the latter is ω-closed in $V[G][H]$.

Suppose now that $x \in p[j(T)]$ in $V[G][H]$. Then by absoluteness $x \in p[j(T)]$ in $M[G][H]$. As T and T^* have the same projection in any $(< \kappa)$-generic extension of V, it follows that $j(T)$ and $j(T^*)$ have the same projection in any $(< \lambda)$-generic extension of M, and therefore $x \in p[j(T^*)]$ in $M[G][H]$. It follows that $x \notin p[S]$, as the existence of a branch through S_x implies the ill-foundedness of $\mathrm{Ult}(M, \langle j(\Sigma(x \restriction n, f \restriction n)) \mid n \in \omega \rangle)$, in contradiction to the above.

Conversely, suppose that $x \notin p[j(T)]$ in $V[G][H]$. Then $x \notin p[j(T^*)]$ in $V[G][H]$ so there is a rank function f on T^*_x. As x belongs to $M[G][H]$ it follows that f also belongs to $M[G][H]$. For $m(e)$ a measure concentrating on some $j(T^*)_{x \restriction n}$, let α_e equal $[f]_{m(e)}$, the ordinal represented by f in $\mathrm{Ult}(M[G][H], m(e))$ (where $m(e)$ has been canonically lifted from M to $M[G][H]$). Then $\langle \alpha_e \mid e \in \omega \rangle$ is an infinite branch through S_x, as desired. $\square$

Lecture 19

Strong absoluteness

The absoluteness principles that we have considered so far refer exclusively to set-generic extensions. The Lévy-Shoenfield absoluteness principle, however, applies to arbitrary extensions. The *strong absoluteness principles* discussed below are in the tradition of Lévy-Shoenfield and impose no genericity requirement on the extensions considered.

By *extension of* V I shall mean a ZFC model V^* which contains V and has the same ordinals as V. This is best formalised by regarding V as a countable transitive model of ZFC and allowing V^* to range over countable transitive ZFC models which contain V and have the same ordinal height as V.

Lévy-Shoenfield absoluteness. Suppose that φ is a Σ_1 formula with real parameters true in an extension of V. Then φ is true in V.

Any consistent generalisation of Lévy-Shoenfield absoluteness must deal with the following two obstacles:

Counterexample 1. There is a Σ_1 formula with parameter from $H(\omega_2)$ which holds in some (set-generic) extension V^* of V but not in V.

Counterexample 2. There is a Σ_1 formula with parameter from $H((2^{\aleph_0})^+)$ which holds in some (ccc set-generic) extension V^* of V but not in V.

Counterexample 1 is witnessed by the formula "ω_1^V is countable". Counterexample 2 is witnessed by the formula "There is a real not in $\mathcal{P}(\omega)^V$".

Let us say that a Σ_1 *absoluteness principle* is a principle asserting the absoluteness of certain Σ_1 formulas with certain parameters with respect to certain extensions of V. Our counterexamples imply that a consistent Σ_1 absoluteness principle must impose some restriction either on the choice of formulas, the choice of parameters, the choice of extensions, or a combination of the three.

I offer three proposals. The first allows arbitrary parameters, at the cost of restricting the choice of extensions. The second allows arbitrary extensions, at the cost of restricting the allowable parameters. And the third weakens the parameter restrictions of the second proposal, at the cost of restricting the choice of formulas in various ways.

a. Σ_1 absoluteness with arbitrary parameters.

A first attempt to avoid Counterexample 1 is to require that V and V^* have the same ω_1. But Σ_1 absoluteness with parameters from $H(\omega_2)$ even for ω_1-preserving extensions is also inconsistent: Let A be a stationary subset of ω_1. Then the formula which asserts that A contains a CUB subset is Σ_1 and true in a cardinal-preserving (set-generic) extension; therefore Σ_1 absoluteness with parameters from $H(\omega_2)$ for ω_1-preserving extensions implies that A

contains a CUB subset. But there are disjoint stationary subsets of ω_1, giving disjoint CUB subsets of ω_1, a contradiction.

Even requiring stationary-preservation at ω_1 (i.e, that stationary subsets of ω_1 in V remain stationary in V^*) results in inconsistency:

Theorem A. There exists an extension V^* of V which is stationary-preserving at ω_1 such that some Σ_1 sentence with parameters from $H(\omega_2)^V$ true in V^* is false in V.

Proof. By a theorem of Beller-David there is an extension V^* with the same ω_1 as V containing a real R such that $L_\alpha[R]$ fails to satisfy ZFC for each ordinal α. Moreover, V^* is stationary-preserving at ω_1. Now suppose that the Theorem fails. Then there is such a real R in V, as this property of R can be expressed by a Σ_1 sentence with parameters R and ω_1. In particular, ω_1 is not inaccessible to reals. It is easy to see that the failure of the Theorem implies that Σ^1_3-absoluteness holds between V and its stationary-preserving at ω_1 extensions. It then follows that ω_1 is inaccessible to reals after all, contradiction. $\square$

One could continue to make further restrictions on the extension V^*, such as stationary-preservation at ω_1 together with full cardinal-preservation, in the hope of achieving the consistency of $\Sigma_1(H(\omega_2))$ absoluteness (without imposing the requirement that V^* be a set-generic extension of V). But we must also reckon with Counterexample 2.

A possible solution is described by the following. I say that an extension V^* of V *strongly preserves* $H(\kappa)$ iff the $H(\kappa)$ of V^* equals the $H(\kappa)$ of V and all cardinals of V less than or equal to Card $(H(\kappa)) = 2^{<\kappa}$ remain cardinals in V^*.

Σ_1 *absoluteness with arbitrary parameters.* Suppose that κ is an infinite cardinal and a Σ_1 formula φ with parameters from $H(\kappa^+)$ holds in an extension V^* of V which strongly preserves $H(\kappa)$. Then φ holds in V.

When κ is ω, this is Lévy-Shoenfield absoluteness. When κ is ω_1, this asserts $\Sigma_1(H(\omega_2))$ absoluteness for extensions which do not add reals and which preserve cardinals up to $2^{\aleph_0}$. Note that in the presence of $\sim$ CH, this axiom does rule out the two standard set-forcings for destroying the stationarity of a subset of ω_1.

It is possible that a weaker restriction on the extension V^* will suffice, provided we insist only on arbitrary *ordinal* parameters.

Σ_1 *absoluteness with arbitrary ordinal parameters.* Suppose that φ is a Σ_1 formula with ordinal parameters which holds in a cardinal-preserving extension of V. Then it holds in V.

Counterexample 1 is avoided as we insist on cardinal-preservation. And Counterexample 2 is avoided as we only allow ordinal parameters.

b. Σ_1 *absoluteness for arbitrary extensions.*

Counterexamples 1 and 2 imply that to obtain a consistent version of absoluteness for arbitrary Σ_1 formulas with respect to arbitrary extensions, we must impose some restriction on our choice of parameters. A suitable restriction is perhaps provided by the following definition.

Definition. Let x belong to V and let V^* be an extension of V. I say that x is *absolute between* V *and* V^* iff there is some parameter-free formula which defines x not only in V but also in V^*.

Σ_1 *absoluteness for arbitrary extensions.* Suppose that V^* is an extension of V and φ is a Σ_1 formula whose parameters are absolute between V and V^*. Then if φ is true in V^* it is also true in V.

Counterexample 1 is avoided as ω_1^V may fail to be absolute between V and extensions in which it is countable. Counterexample 2 is avoided as $\mathcal{P}(\omega)^V$ may fail to be absolute between V and extensions in which new reals are added.

c. Cardinality, cofinality, CUB and powerset absoluteness principles.

Other forms of strong absoluteness result by onsidering special types of Σ_1 formulas. First I generalise our earlier notion of absolute parameter.

Definition. Suppose that x belongs to V, P is a subset of V and V^* is an extension of V. Then x is *absolute relative to parameters in* P *between* V *and* V^* iff there is a formula with parameters from P which defines x not only in V, but also in V^*.

For cardinality and cofinality we have the following absoluteness principles.

Cardinality absoluteness. Suppose that α is an ordinal, V^* is an extension of V and α is absolute relative to bounded subsets of α between V and V^*. Then if α is collapsed (i.e., not a cardinal) in V^*, it is also collapsed in V.

Cofinality Absoluteness. Suppose that α is an ordinal, V^* is an extension of V and α is absolute relative to bounded subsets of α between V and V^*. Then if α is singular in V^*, it is also singular in V.

For largeness in the sense of the CUB filter we have:

CUB absoluteness. Suppose that X is a subset of a regular cardinal κ, V^* is an extension of V and X is absolute relative to ordinals and bounded subsets of κ between V and V^*. If cofinalities at most κ are preserved between V and V^* and X contains a CUB subset in V^*, then it contains one in V.

The following is a strong absoluteness principle for the powerset operation.

Powerset absoluteness. Suppose X is a subset of $P(\kappa)$, κ an infinite cardinal, V^* is an extension of V and X is absolute relative to ordinals and subsets of κ between V and V^*. If cardinals at most κ are preserved between V and V^* then the cardinality of X in V^* equals its cardinality in V.

Lecture 20

The consistency strength of strong absoluteness principles

I do not know if any of the above principles are provably consistent relative to large cardinals. In this subsection I provide some lower bounds on their consistency strength.

Theorem B. Σ_1 absoluteness with arbitrary parameters implies that the GCH fails at every infinite cardinal, and for regular uncountable κ, there is no κ-Suslin tree.

Proof. Suppose that the GCH held at the infinite cardinal κ. Choose $S \subseteq \kappa^+$ to be a fat-stationary subset of κ^+ which does not contain a CUB subset. (S

is *fat-stationary* iff $S \cap C$ contains closed subsets of any ordertype less than κ^+, for each CUB $C \subseteq \kappa^+$.) The existence of such a set is guaranteed by a result of Krueger. Then the forcing P that adds a CUB subset to S using closed subsets of S ordered by end-extension has cardinality κ^+ and, using the fatness of S, is κ^+-distributive. It follows that $H(\kappa^+)$ is strongly preserved by P. But a CUB subset of S witnesses a Σ_1 formula with parameter S not true in the ground model, in contradiction to our hypothesis.

Suppose that there were a κ-Suslin tree T for an uncountable regular cardinal κ. Then forcing with this tree strongly preserves $H(\kappa)$ and adds a witness to a Σ_1 formula with parameter T not witnessed in the ground model, in contradiction to our hypothesis. □

Corollary. Σ_1 absoluteness with arbitrary parameters implies the consistency of a measurable cardinal κ of Mitchell order κ^{++}.

To study Σ_1 absoluteness with arbitrary ordinal parameters we make use of the following result.

Lemma. Suppose that there is no inner model with a measurable cardinal α of Mitchell order α. Suppose that κ is a singular cardinal. Then there is a fat-stationary $S \subseteq \kappa^+$ which is definable with parameter κ in Mitchell's core model K for sequences of measures and does not contain a CUB subset in V.

Corollary. The consistency strength of Σ_1 absoluteness with arbitrary ordinal parameters is at least that of a measurable cardinal κ of Mitchell order κ.

Proof. Assume Σ_1 absoluteness with arbitrary ordinal parameters and that there is no inner model with a measurable cardinal α of Mitchell order α. Let κ be a singular strong limit cardinal. By the previous lemma, there is a fat-stationary $S \subseteq \kappa^+$ in K which does not contain a CUB subset. The forcing P that adds a CUB subset to S using closed subsets of S ordered by end-extension is κ^+-distributive and witnesses a new Σ_1 formula with parameter S. But K is not changed by this forcing and therefore there is a formula with ordinal parameters which defines S both in V and in a P-generic extension. Thus to avoid a counterexample to our absoluteness hypothesis, P must collapse a cardinal over V, which is only possible if the GCH fails at κ. This gives the consistency of a measurable κ of Mitchell order κ^{++}. □

Theorem C. Suppose that Σ_1 absoluteness for arbitrary extensions holds. Then there is an inner model with a measurable cardinal α of Mitchell order α.

Proof. If there is no inner model with a measurable cardinal α of Mitchell order α, then by the lemma, if κ denotes $\aleph_\omega$, there is a fat-stationary subset S of κ^+ which is definable in K with parameter κ and does not contain a CUB subset. Then there is a formula which defines S not only in V but also in $V[G]$, where G is generic for adding a CUB subset to S. This is a violation of our absoluteness hypothesis. □

Theorem D. Cardinal absoluteness implies that for each infinite cardinal κ, κ^+ is greater than (κ^+ of HOD).

Proof. If G is generic for the Lévy collapse of κ^+ to ω, then HOD is the same in V and in $V[G]$, by the homogeneity of the forcing. This contradicts our absoluteness hypothesis. □.

Corollary. Cardinal absoluteness implies that there is an inner model with a strong cardinal, and, if there is a proper class of subtle cardinals, there is an inner model with a Woodin cardinal.

It is possible to extend the Corollary to obtain inner models with a proper class of Woodin cardinals containing any given set, under the assumption of cardinal absoluteness and a proper class of subtle cardinals. This is more than enough to imply Projective Determinacy.

Theorem D also holds for cofinality absoluteness, as the latter implies cardinality absoluteness. CUB and powerset absoluteness have at least the consistency strength of a measurable cardinal α of Mitchell order α using the proof of Theorem C.