Some remarks on a sixty-year-old modeltheoretic method

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joint work with
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Sy-David Friedman *1953

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Roland Fraïssé (1920-1988)

Sur quelques classifications des systèmes de relations

Thesis, Paris, 1953

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Fraïssé (1953) Ehrenfeucht(1961)

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in particular,

$$\underbrace{(\{0, 1, \ldots, 2^m + 1\}, <)}_{\mathcal{A}_m \in \textsc{Even}_<} \;\; \equiv\rangle_{\mathrm{FO}_m} \;\; \underbrace{(\{0, 1, \ldots, 2^m\}, <)}_{\mathcal{B}_m \notin \textsc{Even}_<}$$

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$\mathrm{P} \neq \mathrm{NP}$? $\mathrm{NP} \neq \text{co-NP}$?

via the Ehrenfeucht-Fraïssé method

Content

1. Key problems and the relevant logics.
2. Limitations of the Ehrenfeucht-Fraïssé method.

Key problems and the relevant logics

$P \neq NP?$ $\qquad$ $NP \neq \text{co-}NP?$

$$
\begin{array}{ccc}
\text{LFP} & \approx & \text{P} \\
\Sigma^1_1 & \approx & \text{NP} \\
\Pi^1_1 & \approx & \text{co-NP}
\end{array}
$$

The logic Σ_1^1:

Fragment of second-order logic consisiting of the sentences of the form

$$\exists X_1 \ldots \exists X_\ell \, \psi,$$

where $\psi = \psi(X_1, \ldots, X_\ell) \in \mathrm{FO}$ and $X_1, \ldots, X_\ell$ are second-order variables of any arity.

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Fagin's Theorem. $\Sigma^1_1 = \mathrm{NP}$. The logic Σ^1_1 captures the complexity class NP.

- $\mathrm{NP} \leq \Sigma^1_1$. Every NP-class of structures is axiomatizable in Σ^1_1.
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3-Col (the class of 3-colorable graphs)

$$\exists X \exists Y \exists Z \Big(\forall u (Xu \vee Yu \vee Zu) \wedge \forall u \forall v \big(Euv \rightarrow (\neg(Xu \wedge Xv) \wedge \neg(Yu \wedge Yv) \wedge \neg(Zu \wedge Zv))\big)\Big)$$

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Fact. $\Pi^1_1 = \text{co-NP}$. The logic Π^1_1 captures the complexity class co-NP

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LFP-operator $\approx$ μ-operator of recursion theory

$$[\mathrm{LFP}_{x_1,\dots,x_r,Z}\psi]\, u_1 \dots u_r$$

Z r-ary second-order variable

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$$\begin{aligned} \mathrm{P} \neq \mathrm{NP} &\iff \text{3-}\textsc{Col}_< \notin \mathrm{P} \\ &\iff \text{3-}\textsc{Col}_< \text{ is not axiomatizable in LFP} \\ &\iff \text{there is a } (\text{3-}\textsc{Col}_<, \mathrm{LFP})\text{-sequence.} \end{aligned}$$

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(Ajtai, Fagin) Reachability in directed graphs is not axiomatizable in monadic Σ^1_1.

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LFP

(Grohe) The arity hierarchy of LFP is strict.

(Kubierschky) For $k \in \mathbb{N}$ the hierarchy of LFP formulas of arity at most k whose m-th member consists of formulas with at most m nested fixed-point operators is strict.

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> *It is known that $\Sigma^1_1 \neq \Pi^1_1$ if and only if such a separation can be proven via second-order Ehrenfeucht-Fraïssé games. Unfortunately, "playing" second-order Ehrenfeucht-Fraïssé games is very difficult, and the above promise is still largely unfulfilled; for example, the equivalence between the NP = co-NP question and the $\Sigma^1_1 = \Pi^1_1$ question has not so far led to any progress on either of these questions.*
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Grohe's and Kubierschky's "arity hierarchy results" refer to logics with nonmonadic second-order quantifiers

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No (3-$\textsc{Col}_{<}$*, LFP)-sequence* $(\mathcal{A}_m, \mathcal{B}_m)_{m \in \mathbb{N}}$ *can be generated by an algorithm* $\mathbb{S}$*,*
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EXAMPLE. There is a (3-COL$_<$, FO)-sequence computable in space $O(\log(\|\mathcal{A}_m\| + \|\mathcal{B}_m\|))$.

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Proof sketch.

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1. Let $(\mathcal{A}_m, \mathcal{B}_m)_{m\in\mathbb{N}}$ be a (3-COL$_<$, LFP)-sequence. No polynomial time algorithm $\mathbb{C}$ accepts all $\mathcal{A}_m$ and rejects all $\mathcal{B}_m$.

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2. Assume $\mathbb{G}$ generates $(\mathcal{A}_m, \mathcal{B}_m)_{m \in \mathbb{N}}$ in polynomial output time.

3. We turn $\mathbb{G}$ into a polynomial time algorithm $\mathbb{C}$ such that for infinitely many m,

$$\mathbb{C} \text{ accepts } \mathcal{A}_m \quad \text{and} \quad \mathbb{C} \text{ rejects } \mathcal{B}_m,$$

   that is, there is a increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all m,

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4. An infinite subsequence of a (3-COL$_<$, LFP)-sequence is a (3-COL$_<$, LFP)-sequence; thus, $(\mathcal{A}_{f(m)}, \mathcal{B}_{f(m)})_{m\in\mathbb{N}}$ is a (3-COL$_<$, LFP)-sequence.

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2. Assume $\mathbb{G}$ generates $(\mathcal{A}_m, \mathcal{B}_m)_{m\in\mathbb{N}}$ in polynomial output time.

3. We turn $\mathbb{G}$ into a polynomial time algorithm $\mathbb{C}$ such that for infinitely many m,

$$\mathbb{C} \text{ accepts } \mathcal{A}_m \quad \text{and} \quad \mathbb{C} \text{ rejects } \mathcal{B}_m,$$

   that is, there is a increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all m,

$$\mathbb{C} \text{ accepts } \mathcal{A}_{f(m)} \quad \text{and} \quad \mathbb{C} \text{ rejects } \mathcal{B}_{f(m)}.$$

4. An infinite subsequence of a (3-COL$_<$, LFP)-sequence is a (3-COL$_<$, LFP)-sequence; thus, $(\mathcal{A}_{f(m)}, \mathcal{B}_{f(m)})_{m\in\mathbb{N}}$ is a (3-COL$_<$, LFP)-sequence.

5. Items 3 and 4 contradict 1.

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– If $\mathrm{GI} \in \mathrm{P}$, then no (Q, Σ_1^1)-sequence can be generated in polynomial output time (thus, no $(\textsc{not-3-Col}, \Sigma_1^1)$-sequence can be generated in polynomial output time).